II.7. THE ANALYSIS OF THE SCHRÖDINGER EQUATION BY ELECTRICAL MODELLING

1. Work Purpose

The objective of this paper is to study the behavior of a particle of mass *m*, situated in the potential well $V = V(x)$, whose dynamics is described by the Schrödinger equation. For this we will obtain the energy values of the energy, as well as the corresponding wave functions.

2. Theory

We will use the analogy between the differential equation for the wave function $\Psi(x)$ and the differential equation for the voltage $U = U(x)$, in an electric circuit conveniently chosen.

The proposed electric model is presented in Figure 1.

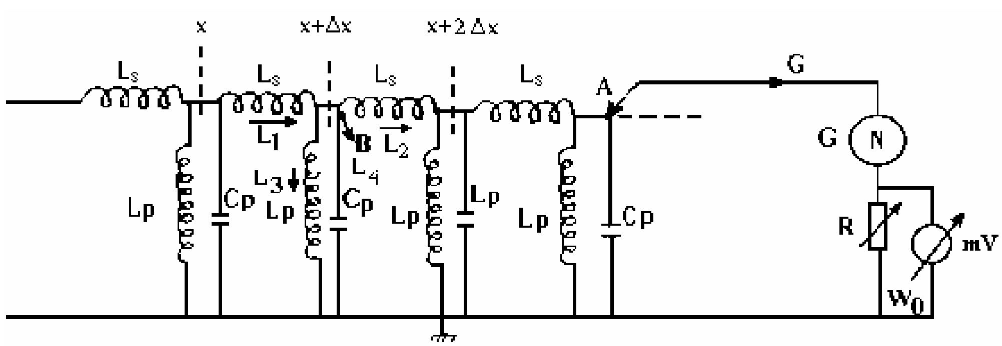

Figure 1.

In this model, L_s and L_p are ideal inductances, C_p is an ideal capacitor, and R is a resistance allowing the measurement of the current i_g through the generator G (of constant voltage and variable frequency), the computing relation being $i_G = U_0 / R$, where the voltage U_0 is read by the millivoltmeter mV.

Let ω be the frequency when the current i_G is minimum (theoretically null if L_s, L_p and C_p are ideal circuit elements). In this case, the point A where G is connected at the circuits could be anywhere and the first Kirchhoff law written for any knot (let B be this one) is

$$I_1 - I_2 - I_3 - I_4 = 0. \qquad (1)$$

The second Kirchhoff law written for the branch traversed by the current I_1 gives:

$$\omega L_s I_1 + U(x) - U(x - \Delta x) = 0 \quad \Rightarrow \quad I_1 + \frac{U(x) - U(x - \Delta x)}{\Delta x}\frac{\Delta x}{\omega L_s} = 0. \qquad (2)$$

Using the Lagrange theorem for $U(x)$, this becomes

$$I_1 \cong -\left.\frac{dU}{dx}\right|_{x - \frac{\Delta x}{2}} \cdot \frac{\Delta x}{\omega L_s}. \qquad (3)$$

In the same way the second Kirchhoff law written for the branch traversed by the current I_2 gives:

$$\omega L_s I_2 + U(x + \Delta x) - U(x) = 0, \qquad (4)$$

or, using the Lagrange theorem,

$$I_1 \cong -\left.\frac{dU}{dx}\right|_{x + \frac{\Delta x}{2}} \cdot \frac{\Delta x}{\omega L_s}. \qquad (5)$$

The currents I_3 and I_4 results directly:

$$I_3 = \frac{U(x)}{\omega L_p}; \quad I_4 = \omega C_p \cdot U(x). \qquad (6)$$

In the end the relation (1) becomes:

$$\left(\left.\frac{dU}{dx}\right|_{x + \frac{\Delta x}{2}} - \left.\frac{dU}{dx}\right|_{x - \frac{\Delta x}{2}}\right) \cdot \frac{\Delta x}{\omega L_s} - \left(\frac{1}{\omega L_p} - \omega C_p\right) \cdot U(x) = 0. \qquad (7)$$

Using again the Lagrange theorem, this time for $\frac{dU}{dx}$, we have:

$$\left.\frac{dU}{dx}\right|_{x+\frac{\Delta x}{2}} - \left.\frac{dU}{dx}\right|_{x-\frac{\Delta x}{2}} \cong \left.\frac{d^2U}{dx^2}\right|_{x} \cdot \Delta x . \tag{8}$$

By replacing (8) in (7), it results that the differential equation that is satisfied by the voltage $U = U(x)$ (the voltage being measured between the knot and the ground) is:

$$\frac{d^2U}{dx^2} - \frac{L_s}{(\Delta x)^2}\left(\frac{1}{L_p} - \omega^2 C_p\right)U = 0 . \tag{9}$$

The relation (9) is similar with the Schödinger equation:

$$\frac{d^2\Psi}{dx^2} - \frac{2m}{\hbar^2}(V - E)\Psi = 0 . \tag{10}$$

In order to have the same numerical values (that is $U(x) = \Psi(x)$), it is necessary that the boundary conditions are the same and the following equality must be satisfied:

$$\frac{L_s}{(\Delta x)^2}\frac{1}{L_p} = \frac{2m}{\hbar^2}V , \tag{11.a}$$

$$\frac{L_s}{(\Delta x)^2}\omega^2 C_p = \frac{2m}{\hbar^2}E . \tag{11.b}$$

We choose to model $V = V(x)$ by applying $L_p = L_p(x)$; we obtain:

$$L_p(x) = \frac{L_s}{(\Delta x)^2}\frac{\hbar^2}{2m}\frac{1}{V(x)}, \tag{12.a}$$

$$E = \frac{1}{(\Delta x)^2}\frac{\hbar^2}{2m}\left(\frac{\omega}{\omega_0}\right)^2 , \tag{12.b}$$

where $\omega_0 = (L_s C_p)^{-1/2}$. Then, by measuring the frequency ω for $i_G = 0$, we can compute the energy values by means of the relation (12.b).

In the following we will analyze two potential well configurations, the infinite (Fig 2.a) and the finite (Fig 2.b) ones.

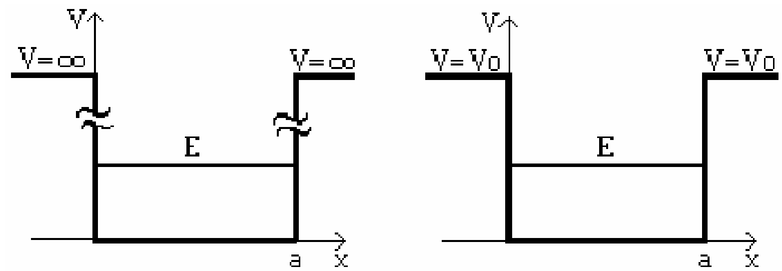

Figure 2.

We specify that, experimentally, the condition $V = \infty$ implies $L_p = 0$ (that is equivalent with a ground connection); the condition $V = 0$ implies $L_p = \infty$ (that is equivalent with the absence of L_p - open circuit). The case $V = V_0$ has been particularized by taking $L_p = L_s$.

3. The experimental set-up

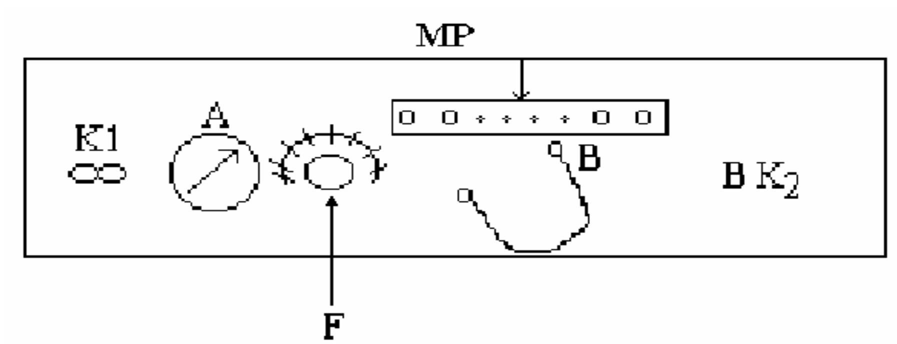

Figure 3.

By maintaining the basic idea presented in Figure1, we realised an experimental set-up with the front panel presented in the Figure 3:

1. K_1 is a switch that lets us use the instrument A as a minimum values indicator (the position "$\Psi(x)$"). **Attention!** The instrument has the zero in the middle of the scale.
2. F is a potentiometer that allows the calibration and the reading of the frequency supplied by the generator.

3. M.P. represents the measuring points, marked in the superior part of the panel with figures from 1 to 20. In the case of our set-up, the interval $x \in (0, a)$ has been simulated with 10 cells (L_s, L_p, C_p), so that we have $\frac{a}{\Delta x} = N = 10$. The correspondence between the figures attached to the measuring points and the x values is given in Table 1.
4. B is a jack that allows the reading of the voltage $U = U(x)$ respectively that of the wave function $\Psi = \Psi(x)$ by applying it on the desired M.P. (with K_1 in the position "$\Psi(x)$").
5. K_2 is a switch that allows us to obtain the configuration from Figure 2.a (when in the position "∞") and 2.b (when in the position "V_0"), respectively.

Table 1

M. P.	1	2	3	4	5	6	7	8	9	10
x/a	– 0.4	– 0.3	– 0.2	– 0.1	0.0	0.1	0.2	0.3	0.4	0.5
M. P.	11	12	13	14	15	16	17	18	19	20
x/a	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5

4. Working procedure

a) Infinite potential well

We switch K_2 in the position "∞" and K_1 in the position "E". Starting with F from the minimum frequency, we slowly increase the frequency until A indicates the first minimum corresponding to the fundamental energy state; we write down the corresponding frequency, ω_1. By switching K_1 in the position "$\Psi(x)$", we touch with B the measuring points corresponding to the interval $(0, a)$; then we fill the first line in Table 2 with the values of $U_1 = U_1(x/a)$. After that, we switch back K_1 in the position “E” and we increase the frequency until the next minimum, repeating the above measurements.

Table 2.

Case	n	x/a	– 0.4	...
a)	1	U_1 (V)	...	...
	2	U_2 (V)	...	...
b)	1	U_1 (V)	...	...
	2	U_2 (V)	...	...

b) Finite potential well

We switch K_2 in the position “V_0” and we repeat the operations indicated at the point *a)*; we write down the values of the frequency and the voltage $U_2 = U_2(x/a)$ for the first two minima.

5. Experimental data processing

a) Infinite potential well

In this case the equation (10) has the solution:

$$|\Psi_n(x)|^2 = \sqrt{\frac{2}{a}} \sin^2\left(n\pi \frac{x}{a}\right), \; n = 1, 2, 3, ... \tag{13}$$

and the energy values are given by the relation:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}. \tag{14}$$

For each frequency we represent the graphs of the dependence:

$$\frac{a}{2}\Psi_n(x) \equiv \psi_n\left(\frac{x}{a}\right) = \sin^2\left(n\pi \frac{x}{a}\right) \tag{15}$$

and

$$\frac{U_n(x)}{U_{n\max}} \equiv u_n\left(\frac{x}{a}\right), \tag{16}$$

respectively. In Eq. (16), $U_{n\max}$ (n = 1, 2) is the maximum measured value and is extracted from Table 2. The mentioned dependencies will be represented in the same graphic in order to check the correctness of the hypotheses used for the modelling.

Taking into consideration the case of one electron, we calculate the energy values in the two states through the theoretical relation (14) and also through the experimental relation (12b). Then we compare the results. The values of the constants are: $a = 10^{-9}$m; $\Delta x = 10^{-10}$m; $m = 9.1 \cdot 10^{-31}$kg; $\hbar = 1.05 \cdot 10^{-34}$ J·s; $\nu_0 = \omega_0 / 2\pi = 500$ Hz.

b) Finite Potential well

The equation (10) has the solution:

$$\Psi(x) = \begin{cases} A \exp(qx) & \text{for } x < 0 \\ B\left[\cos(kx) + \frac{q}{k}\sin(kx)\right] & \text{for } 0 \le x \le a \\ C \exp(-qx) & \text{for } x > a \end{cases}, \qquad (17)$$

where $q^2 = \frac{2m}{\hbar^2}(V_0 - E)$, $k^2 = \frac{2mE}{\hbar^2}$ The energy values are the solution of the equation:

$$\tan(k\,q) = \frac{2\,k\,q}{k^2 - q^2}. \qquad (18)$$

For the analysed system, Equation (18) admits the following solutions: $E_1/V_0 = 0.068$, $E_2/V_0 = 0.269$ and $E_3/V_0 = 0.962$. From the relations (12.a) and (12b) we obtain the general expression:

$$\frac{E_n}{V_0} = \left(n\frac{\omega}{\omega_0}\right)^2. \qquad (19)$$

We represent graphically the dependency of $\left|U_n(x)/U_{n\max}\right|^2$ on x/a for the two frequencies ω_1, ω_2. We remark the exponential decrease of the wave function modelled through the dependence $U_n(x/a)$ for $x < 0$ and $x > a$.