III. CONDENSED MATTER PHYSICS

III.1. THE SKIN EFFECT

1. Work purpose

The determination of the penetration depth and of the electrical conductivity of a metal.

2. Theory

The skin effect takes place when the electromagnetic waves pass through a conductor that both absorbs and disperses the electromagnetic waves. Due to this effect the current density increases in the surface layers.

The absorption is a phenomenon that takes place during the wave propagation through a dissipative medium and it means the decrease of the wave intensity when the covered distance increases. The metals are dissipative media for the electromagnetic waves. Their intensity rapidly decreases whit the distance, due to the conduction electrons that, under the external alternate field, determine a supplementary electric field inside the conductor. This field overlaps with the external one, weakening it.

We shall consider the electromagnetic wave intensity I_0, at normal incidence on the upper surface of the dissipative metal (see Figure 1). We have to compute the wave intensity $I(z)$ after covering the distance z. We quote $dI(z)$ the wave intensity decrease after covering the infinitesimal distance dz (z, $z+dz$). This decrease is proportional with both $I(z)$ and z:

$$dI(z) = -\alpha I(z) \cdot dz , \qquad (1)$$

α being the absorption coefficient and the minus sign showing that the intensity decreases when the absorbent layer increases.

To find out the wave intensity at a certain distance z we shall make the sum of all the variations $dI(z)$. So that we integrate the relation (1)

$$\int_{I_0}^{I(z)} \frac{dI(z)}{I(z)} = -\alpha \int_0^z dz \qquad (2)$$

and we obtain:

$$I(z) = I_0 \exp(-\alpha z). \qquad (3)$$

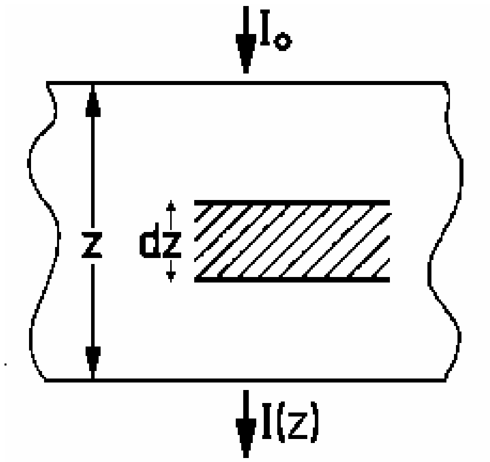

Figure 1.

The relation (3) shows that in a conductor the intensity of the electromagnetic waves exponentially drops with the distance. As $I \propto E^2$, the amplitude $E(z)$ also drops exponentially with the distance:

$$E(z) = E_0 \exp\left(-\frac{\alpha z}{2}\right) = E_0 \exp\left(-\frac{z}{2\delta}\right), \qquad (4)$$

where E_0 is the electric field amplitude of the incident wave and $\delta = 1/\alpha$, called penetration depth or skin thickness,, represents the distance over which the wave intensity decreases *e* times. This depth depends on the wave frequency ν and on the medium electric and magnetic properties.

Let us consider an infinite conductor half-space. We choose a Cartesian system of coordinate axes, such that the *Ox* and *Oy* axes belong to the separation plane (*z* = 0), and the *Oz* axis is oriented towards the interior of the conductor (see Figure 2). The electric field intensity vector $\vec{E}$ and the conduction current density vector $\bar{j}$ are oriented parallel to the

Ox axis, and the magnetic field induction vector $\vec{B}$ is oriented parallel to the *Oy* axis. Hence $\vec{E} = (E_x, 0, 0)$, $\vec{j} = (j_x, 0, 0)$ and $\vec{B} = (0_y, B, 0)$. The vector components are functions of the coordinate *z* and of the time *t* (these components do not vary with the *x* and *y* coordinates). The equations that govern the skin effect analysis are the equations of the electromagnetic wave propagation in substances.

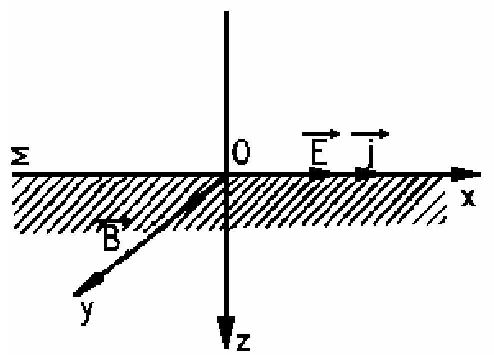

Figure 2.

Electromagnetic wave propagation through a conductor is studied by taking into account that the conduction current density is much greater than the displacement current density. Neglecting the displacement current, we obtain the following equations for the electric and magnetic field propagation through the conductors:

$$\Delta\vec{E} = \sigma\mu\frac{\partial\vec{E}}{\partial t}, \qquad \Delta\vec{B} = \sigma\mu\frac{\partial\vec{B}}{\partial t} \tag{5}$$

In our case, we have:

$$\frac{\partial^2 E_x}{\partial z^2} = \sigma\mu\frac{\partial E_x}{\partial t}, \qquad \frac{\partial^2 B_y}{\partial z^2} = \sigma\mu\frac{\partial B_y}{\partial t}. \tag{6}$$

We remark that we may write the *y* component of the magnetic field $\vec{B}$ as a function of the *x* component of the electric field $\vec{E}$, if we use the Maxwell-Faraday equation, which can be rewritten as follows:

$$\left(\nabla \wedge \vec{E}\right)_y = \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}. \tag{7}$$

We accept a periodical time variation of the electric field, current density and magnetic field such that:

$$E_x(z,t) = E(z)\exp(i\omega t),\ j_x(z,t) = j(z)\exp(i\omega t),\ B_y(z,t) = B(z)\exp(i\omega t). \tag{8}$$

Replacing the expression of E_x from Eq. (8) in Eq. (6), we obtain:

$$\frac{d^2 E(z)}{dz^2} = i\omega\sigma\mu E(z). \tag{9}$$

Quoting $p^2 = i\omega\sigma\mu$, the differential equation (9) will become:

$$\frac{d^2 E(z)}{dz^2} = p^2 E(z). \tag{10}$$

The general solution of this differential equation is:

$$E(z) = A_1 \exp(pz) + A_2 \exp(-pz), \tag{11}$$

where A_1 and A_2 are two integration constants and

$$p = \sqrt{i\omega\sigma\mu} = \sqrt{i}\sqrt{\omega\sigma\mu} = (1+i)\sqrt{\frac{\omega\sigma\mu}{2}}, \tag{12}$$

where we have used $\sqrt{i} = (1+i)/\sqrt{2}$. The integration constants are determined from the following conditions:

- for $z \to +\infty, E(z) \to 0$ hence $A_1 = 0$;
- for $z \to 0, E(z) = E(0) \equiv E_0 = A_2$.

So, the solution (11) will become:

$$E(z) = E_0 \exp(pz) = E_0 \exp\left[-(1+i)\sqrt{\frac{\omega\sigma\mu}{2}} \cdot z\right]. \tag{13}$$

If we introduce the constant:

$$\delta = \frac{1}{\sqrt{2\omega\sigma\mu}}, \tag{14}$$

the equation (13) may be written as:

$$E(z) = E_0 \exp\left[-(1+i)\frac{z}{2\delta}\right] = E_0 \exp\left(-\frac{z}{2\delta}\right)\cdot\exp\left(-i\frac{z}{2\delta}\right). \qquad (15)$$

Taking into account the equation (8) and (15) we obtain then:

$$E_x(z,t) = E_0 \exp\left(-\frac{z}{2\delta}\right)\exp\left[i\left(\omega t - \frac{z}{2\delta}\right)\right]. \qquad (16)$$

To compute the current density j_x and the magnetic field B_y, we use similar relations:

$$j_x(z,t) = \sigma E_0 \exp\left(-\frac{z}{2\delta}\right)\exp\left[i\left(\omega t - \frac{z}{2\delta}\right)\right], \qquad (17)$$

$$B_y(z,t) = \frac{2\delta\sigma\mu}{1+i} E_0 \exp\left(-\frac{z}{2\delta}\right)\exp\left[i\left(\omega t - \frac{z}{2\delta}\right)\right]. \qquad (18)$$

In these relations,

$$\delta = \frac{1}{\sqrt{2\omega\sigma\mu}} = \frac{1}{\sqrt{4\pi\upsilon\sigma\mu}} \qquad (19)$$

represents the penetration depth of the electromagnetic wave through the conductor. Its value varies inversely proportional with the square root of both the frequency of the field ν and the metal conductivity σ. Due to this relation, we may notice that, simultaneously with the absorption, there is a dispersion of the electromagnetic waves. If we increase the frequency ν, the penetration depth decreases, meaning that the electromagnetic wave is localized at the conductor surface. Due to this effect, the conductors used for high frequency currents may look like pipes, to save up material.

In Table 1 we find the values of the penetration depth δ for an alternate current through a copper conductor, for two frequencies of the alternate current: 50 Hz and $5 \cdot 10^5$ Hz. Hence the skin thickness δ decreases with the increase of the current frequency.

Table 1

Material	σ ($\Omega^{-1}\cdot m^{-1}$)	ν (Hz)	δ (mm)
Copper	$5.8\cdot 10^7$	50	4.67
Copper	$5.8\cdot 10^7$	$5\cdot 10^5$	0.0467

In this paper we will determine the penetration depth δ and the electrical conductivity σ for various frequency values. An electromagnetic wave of a known frequency will fall on a conductor made from one or more metallic plates and we will record the amplitude of the alternate voltage determined in a receiving coil by the waves that pass through the plates. Due to the proportionality between the voltage and the electric field intensity, the amplitude of the incident alternate voltage U_0 drops exponentially with the distance *z* and it is described by Eq. (4), meaning:

$$U(z) = U_0 \exp\left(-\frac{z}{2\delta}\right), \tag{20}$$

so that

$$\log U = \log U_0 - \frac{z}{2\delta}, \tag{21}$$

or

$$\log \frac{U_0}{U} = \frac{z}{2\delta}. \tag{22}$$

If we draw the dependence of $\log U_0/U$ upon *z*, we will obtain a straight line with the slope $m = 1/2\delta$. By determining the slope we will find the penetration depth $\delta = 1/2m$.

From the relation (14), we notice that δ is a linear function of $1/\sqrt{\nu}$

$$\delta = \frac{1}{\sqrt{4\pi\sigma\mu}} \cdot \frac{1}{\sqrt{\nu}} = m' \frac{1}{\sqrt{\nu}}. \tag{23}$$

Drawing the dependence of δ upon $1/\sqrt{\nu}$, and determining the slope *m'* of the straight line, we will compute the electrical conductivity:

$$\sigma = \frac{1}{4\pi\mu m'^2}, \tag{24}$$

where $\mu \approx \mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{N \cdot A^{-2}}$.

3. Experimental set-up

The experimental set-up (see Figure 3) is made of a sine oscillation generator in the frequency range 10 – 100 kHz with a voltage level of 1000 mV (Versatester – type E0502), on which we connect an oscillator coil B_1 and a receiving coil B_2. Between them we put some metallic sheets (of Cu, Al, Sn). By supplying an alternate current (through the coaxial cable C_1) to the B_1 coil, a phenomenon of electromagnetic induction appears, inducing an alternate voltage in the receiving coil B_2. The induction current frequency may be varied, and the induced voltage in B_2 is measured with the apparatus millivoltmeter (mV).

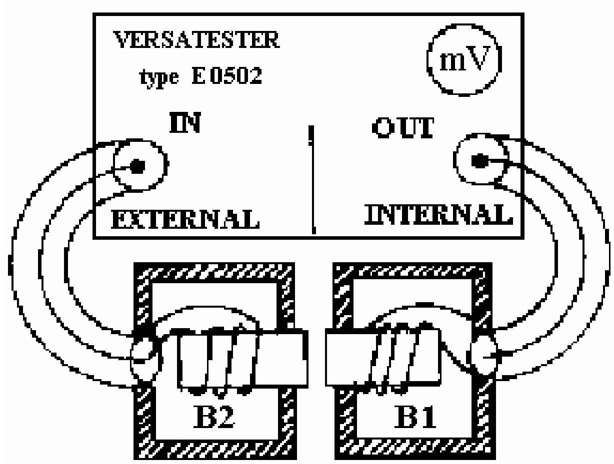

Figure 3.

4. Working procedure

1. The device is plugged in at 220 V a. c. and the coaxial cable C_1 of the oscillator coil B_1 is connected at the muff "IESIRE 50". We press the key

"10 – 100 kHz", and the level selector (NIVEL INTERN) is on "1000 mV".

2. With the fine frequency selector "FRECVENTA" we choose a frequency (for instance 50 kHz), which is read on the digital display, by choosing "INTERN F" from the internal switch.

3. We determine on the voltmeter the voltage U_0 for zero absorber thickness, switching the external switch on "EXTERN F".

4. We choose a metallic sheet of a known thickness z (z_{Al} = 80 μm, z_{Cu} = 40 μm, z_{Sn} = 50 μm), which is placed between the coils B_1 and B_2, and we determine the received voltage on the millivoltmeter, switching the external switch on "EXTERN F". We successively add sheets of the same metal and we record for every total thickness $z' = n \cdot z$ (n being the total number of sheets), the received voltage. At least another 4 values of the frequency (for instance ν = 60, 70, 80, and 90 kHz) must be used for all thicknesses. The obtained data will be filled in Table 2:

Table 2

Metal	ν (kHz)	z (mm)	U (mV)	U_0/U	log U_0/U	δ (mm)
...	...	...	...	...	...	...

5. We shall repeat the experiment for the other metals too. For each metal, we will fill the obtained data in Table 2.

6. In order to find the electrical conductivity σ of the used materials, after data processing, Table 3 is filled in:

Table 3

Metal	ν (kHz)	$\nu^{1/2}$ ($kHz^{1/2}$)	δ (mm)
...	...	...	...

5. Experimental data processing.

1. For each metal and fixed frequency ν_1, ν_2, ..., we draw on the same diagram the dependency $\log U_0/U = f(z)$, where U_0 is the voltage when

all metallic sheets are removed. Using the relation (22) we obtain, for each metal, a family of straight lines for which we will determine their slopes m_1, m_2, ... We compute the penetration depths δ_1, δ_2, ..., knowing that $\delta = 1/2m$.

2. Using the data from Table 3, another graph is drawn, with the dependency of δ upon $1/\sqrt{\nu}$. A straight line of slope m' is obtained for each metal and using the relation (24) the electrical conductivities σ are determined.

3. In order to determine the slope of the line $y = ax$, we can apply the least square method. Then, the estimated value for the slope is:

$$\bar{a} = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}, \tag{25}$$

where n is the number of pairs $\{x_i, y_i\}$ experimentally measured. The value of the parameter a is affected by the mean square deviation:

$$s_{\bar{a}} = \left\{\frac{\sum_{i=1}^{n}(y_i - \bar{a}x_i)^2}{(n-1)\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]}\right\}^{1/2}. \tag{26}$$

4. The linear dependence between $\log U_0/U$ and z is given by the relation $\log U_0/U = z/2\delta$. We quote $y = \log U_0/U$, $x = z$, $a = 1/2\delta$. The unknown a must be expressed as a function of its estimated value and mean square deviation

$$a = \bar{a} \pm s_{\bar{a}}. \tag{27}$$

Similarly

$$\delta = \bar{\delta} \pm s_{\bar{\delta}}. \tag{28}$$

Taking into account the relation $\delta = \delta(a)$, we have:

$$s_{\bar{\delta}}^2 = \left(\frac{d\delta}{da}\right)_{a=\bar{a}}^2 s_{\bar{a}}^2. \tag{29}$$

5. We apply the same method for the determination of the conductivity σ. In this case $y = \delta$, $x = 1/\sqrt{\nu}$ and

$$\sigma = \bar{\sigma} \pm s_{\bar{\sigma}}. \tag{30}$$