SOLVED PROBLEMS
IN
QUANTUM PHYSICS

Dr. Ing. Valerică D. Ninulescu

Contents

Glossary of notations

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}^*$	the set of non-negative integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\sim$	roughly similar; poorly approximates
$\equiv$	identically equal to
e	exponential function, $\mathrm{e}^z = \exp z$
i	complex unity, $\mathrm{i}^2 = -1$
Re	real part of a complex number
Im	imaginary part of a complex number
c.c.	complex conjugate of the expression in front of c.c.
z^*	complex conjugated of z
$\arg z$	the argument of z; the angle in the polar form of the complex number z
$\boldsymbol{v}$	vector $\boldsymbol{v}$
$\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$	unit vectors in Cartesian coordinate system
$\langle . \rangle$	average
$\int_{\boldsymbol{r}}$	integral over the whole three-dimensional space
$\hat{A}$	operator A
$\hat{A}^\dagger$	adjoint of $\hat{A}$
k_B	Boltzmann constant
CM	centre of mass
lhs	left-hand side
rhs	right-hand side
n-D	n-dimensional
TDSE	time-dependent Schrödinger equation
TISE	time-independent Schrödinger equation
CSCO	complete set of commuting observables
MATLAB	registered trademark of MathWorks, Inc.

Chapter 1

The experimental foundations of quantum mechanics

1.1 Useful Equations

Stefan–Boltzmann law

The *radiant exitance* (energy radiated from a body per unit area per unit time), M, of a blackbody at temperature T grows as T^4:

$$M(T) = \sigma T^4, \tag{1.1}$$

where $\sigma \approx 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$ is Stefan–Boltzmann constant.

Stephan–Boltzmann law can be extended to a grey body of emittance ε as

$$M(T) = \varepsilon\sigma T^4. \tag{1.2}$$

Wien's displacement law

Wien's displacement law states that the wavelength for maximum emissive power from a blackbody is inversely proportional to its absolute temperature,

$$\lambda_{\mathrm{max}} T = b. \tag{1.3}$$

The constant b is called *Wien's displacement constant* and its value is $b \approx 2.898 \times 10^{-3}\,\mathrm{m\,K}$.

Rayleigh–Jeans radiation formula

The spectral energy density of a blackbody is

$$\rho_\nu(\nu, T) = \frac{8\pi\nu^2}{c^3}\, k_\mathrm{B}T, \tag{1.4}$$

The spectral energy density in terms of wavelength can be derived as

$$\rho_\lambda(\lambda, T) = \frac{8\pi}{\lambda^4}\, k_\mathrm{B}T. \tag{1.5}$$

The Rayleigh–Jeans formula agrees with experimental results only at very long wavelengths at any given temperature.

Planck radiation formulas

The spectral energy density of a blackbody is

$$\rho_\nu(\nu, T) = \frac{8\pi\nu^2}{c^3}\,\frac{h\nu}{\exp(h\nu/k_\mathrm{B}T) - 1}. \tag{1.6}$$

The spectral energy density in terms of wavelength can be derived as

$$\rho_\lambda(\lambda, T) = \frac{8\pi hc}{\lambda^5}\,\frac{1}{\exp(hc/\lambda k_\mathrm{B}T) - 1}. \tag{1.7}$$

The spectral radiant exitance M_λ (emitted power per unit of area and unit of wavelength) of a blackbody is

$$M_\lambda(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\,\frac{1}{\exp(hc/\lambda k_\mathrm{B}T) - 1}. \tag{1.8}$$

Einstein equation of the photoelectric effect

$$(1/2)mv_\mathrm{max}^2 = h\nu - \Phi. \tag{1.9}$$

At the incidence of photons of frequency ν, the stopping voltage of the photoelectrons is

$$U_0 = (h/e)\nu - \Phi/e. \tag{1.10}$$

Planck-Einstein relations

$$E = h\nu = \hbar\omega \quad \text{and} \quad \boldsymbol{p} = \hbar\boldsymbol{k}. \tag{1.11}$$

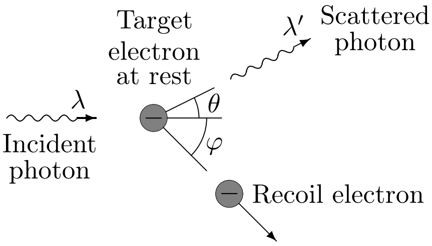


Fig. 1.1 In Compton scattering an x-ray photon of wavelength λ collides with an electron initially at rest. The photon scattered in the direction θ has the wavelength $\lambda' > \lambda$.

Compton effect

Given that λ is the wavelength of the incident photon and λ' that of the scattered photon in the direction θ (angle between the direction of incidence and direction of scattering, see Fig. 1.1), the change in wavelength is

$$\lambda' - \lambda = \lambda_C(1 - \cos\theta), \quad (1.12)$$

where

$$\lambda_C = h/mc \approx 2.426 \times 10^{-12}\,\text{m} \quad (m = \text{electron mass}) \quad (1.13)$$

is called *Compton wavelength of the electron.*

1.2 Questions and problems

1.1 Explain the cooling of Earth surface at night. When is the Earth surface coldest?

Answer. The Earth emits radiation all the time and receives radiation from the Sun during the day only. During a clear day, the Earth surface is heated up; it means that the flux of energy received from the Sun is greater than that emitted by the Earth. Once the Sun goes down, less and less power is received by the Earth. The Earth surface begins to cool and during the night the temperature is contiuously decreasing. The Earth's surface is coldest in the early morning hours.

1.2 Determine the energy of a photon and the number of photons emitted per second by a $P = 2\,\text{mW}$ He–Ne laser that operates on the wavelength $\lambda = 632.8\,\text{nm}$. Interpret the results.

Solution. The energy of a photon is

$$E = h\frac{c}{\lambda} = 3.14 \times 10^{-19}\,\text{J},$$

which is a very small energy for the macroscopic world. The appropriate energy unit at the atomic scale is the electron volt. The energy of a photon emitted by the He–Ne laser is

$$E = 3.14 \times 10^{-19}\,\mathrm{J} = \frac{3.14 \times 10^{-19}}{1.6 \times 10^{-19}}\,\mathrm{eV} = 1.96\,\mathrm{eV}.$$

The number of photons emitted per second is

$$\frac{P}{hc/\lambda} = 6.37 \times 10^{15}\,\mathrm{photons/s}.$$

The huge number of photons explains why we do not notice the quantum nature of the electromagnetic field in everyday life: adding or subtracting one photon does not make a noticeable difference.

1.3 The temperature of a person skin is $\theta_{\mathrm{skin}} = 35\,^\circ\mathrm{C}$.

(a) Determine the wavelength at which the radiation emitted from the skin reaches its peak.

(b) Estimate the net loss of power by the body in an environment of temperature $\theta_{\mathrm{environment}} = 20\,^\circ\mathrm{C}$. The human skin has the emittance $\varepsilon = 0.98$ in infrared and the surface area of a typical person can be estimated as $A = 2\,\mathrm{m}^2$.

(c) Estimate the net loss of energy during one day. Express the result in calories by use of the conversion relation $1\,\mathrm{cal} = 4.184\,\mathrm{J}$.

Solution. (a) By use of Eq. (1.3),

$$\lambda_{\mathrm{max}} = \frac{b}{T_{\mathrm{skin}}} \approx 9.5\,\mu\mathrm{m}.$$

This wavelength is in the infrared region of the spectrum.

(b) The power emitted by the body is [see Eq. (1.2)]

$$P_{\mathrm{emitted}} = \varepsilon\sigma T_{\mathrm{skin}}^4 A \approx 974\,\mathrm{W},$$

while the power absorbed from the environment is

$$P_{\mathrm{absorbed}} = \varepsilon\sigma T_{\mathrm{environment}}^4 A \approx 819\,\mathrm{W}.$$

The net outward flow of energy is

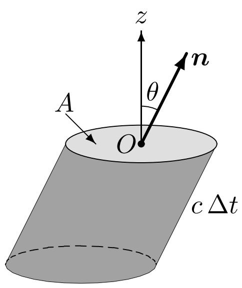
$$P = P_{\mathrm{emitted}} - P_{\mathrm{absorbed}} \approx 155\,\mathrm{W}.$$

(c) The net loss of energy in a time $t = 24 \times 3\,600\,\mathrm{s}$ is $E = Pt \approx 3\,207\,\mathrm{kcal}$. The result is an overestimation of the real net loss of energy, because the clothes we wear contribute to a significant reduction of the skin emittance.

1.4 Prove that the relation between radiant exitance (emitted power per unit area) M of a blackbody and the energy density ρ of the blackbody cavity is

$$M = (1/4)c\rho. \tag{1.14}$$

Solution. Let us consider a small surface of the body; this can be considered as plane. Let us denote by A the area. We choose a 3-D Cartesian coordinate system with the origin O on the emitting surface and the Oz-axis perpendicular to the surface and directed outward (see figure below).



We first write the flux of energy through the surface of area A inside the solid angle $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ around the direction $\boldsymbol{n}$ determined by polar angles θ and φ. During the time interval Δt, the energy emitted inside the solid angle $\mathrm{d}\Omega$ is located inside the cylinder of generatrix paralell to $\boldsymbol{n}$ and length $c\,\Delta t$. The energy inside this volume is $\rho A c\,\Delta t\cos\theta$ and only the fraction $\mathrm{d}\Omega/4\pi$ propagates in the considered solid angle.

The total energy emitted through the surface of area A in the time interval Δt is

$$\int_{\theta=0}^{\theta=\pi/2}\int_{\varphi=0}^{\varphi=2\pi} \rho A c\,\Delta t\cos\theta\,\frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{4\pi} = \frac{1}{4}c\rho A\,\Delta t.$$

It follows that the energy emitted in unit time by unit area is

$$M = (1/4)c\rho.$$

Remark. In case we are interested in the relation between the spectral quantities M_λ and ρ_λ, we restrict ourselves to the radiation in the wavelength interval $(\lambda, \lambda + \mathrm{d}\lambda)$. The energy emitted with the wavelength in the specified interval in Δt is

$$\int_{\theta=0}^{\theta=\pi/2}\int_{\varphi=0}^{\varphi=2\pi} \rho_\lambda \mathrm{d}\lambda\, A c\,\Delta t\cos\theta\,\frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{4\pi} = \frac{1}{4}c\rho_\lambda\,\mathrm{d}\lambda\,A\,\Delta t.$$

The energy emitted in unit time through the unit area of surface is

$$M_\lambda \mathrm{d}\lambda = (1/4)c\rho_\lambda\,\mathrm{d}\lambda$$

from which the relation

$$M_\lambda = (1/4)c\rho_\lambda.$$

is inferred.

1.5 (a) Use Eq. (1.6) to determine the radiant energy density $\rho(T)$ in a cavity whose walls are kept at the temperature T. *Hint*:

$$\int_0^\infty \frac{x^3}{\exp x - 1}\,\mathrm{d}x = \frac{\pi^4}{15}.$$

(b) Use $\rho(T)$ determined above to deduce the Stefan–Boltzmann law.

Solution. (a)

$$\rho(T) = \int_0^\infty \rho_\nu(\nu)\,\mathrm{d}\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{\exp(h\nu/k_\mathrm{B}T) - 1}\,\mathrm{d}\nu$$
$$= \frac{8\pi h}{c^3}\left(\frac{k_\mathrm{B}T}{h}\right)^4 \int_0^\infty \frac{x^3}{\exp x - 1}\,\mathrm{d}x = \frac{8\pi^5 k_\mathrm{B}^4}{15c^3h^3}T^4 \propto T^4.$$

(b) By use of Eq.(1.14) and $\rho(T)$ from above, we have

$$M = \frac{1}{4}c\rho = \frac{2\pi^5 k_\mathrm{B}^4}{15c^2h^3}T^4 = \sigma T^4, \quad \text{where} \quad \sigma = \frac{2\pi^5 k_\mathrm{B}^4}{15c^2h^3} \approx 5.67\times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$$

is Stefan–Boltzmann constant.

1.6 Derive Wien's displacement law from Eq. (1.7).

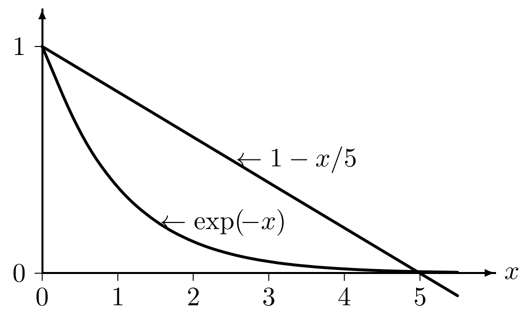
Solution.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\rho_\lambda(\lambda) = \frac{40\pi hc}{\lambda^6}\,\frac{\exp(hc/\lambda k_\mathrm{B}T)}{[\exp(hc/\lambda k_\mathrm{B}T) - 1]^2}\left[-1 + \frac{1}{5}\,\frac{hc}{\lambda k_\mathrm{B}T} + \exp\!\left(-\frac{hc}{\lambda k_\mathrm{B}T}\right)\right].$$

The condition $\mathrm{d}\rho_\lambda(\lambda)/\mathrm{d}\lambda = 0$ yields the transcendental equation

$$1 - x/5 = \exp(-x),$$

where the shorthand notation $x = hc/\lambda k_\mathrm{B}T$ have been used. Besides the trivial solution $x = 0$, a positive solution exists, $x \approx 4.965$ (see figure below).



Graphical solution of the equation $1 - x/5 = \exp(-x)$.

The spectral energy density is maximum for the wavelength $\lambda_{\max}$ given by

$$\lambda_{\max} T \approx \frac{hc}{4.965\, k_{\mathrm{B}}} \approx 2.898 \times 10^{-3}\,\mathrm{m\,K}.$$

1.7 Derive the form of the Planck radiation formula in the limit case:
(a) $hc/\lambda k_{\mathrm{B}}T \ll 1$ (large wavelengths);
(b) $hc/\lambda k_{\mathrm{B}}T \gg 1$ (small wavelengths).
Interpret the results.

Solution. (a)

$$\begin{aligned}\rho_\lambda(\lambda, T) &= \frac{8\pi hc}{\lambda^5}\,\frac{1}{\exp(hc/\lambda k_{\mathrm{B}}T) - 1}\\ &= \frac{8\pi hc}{\lambda^5}\,\frac{1}{\left[1 + \dfrac{hc}{\lambda k_{\mathrm{B}}T} + \left(\dfrac{hc}{\lambda k_{\mathrm{B}}T}\right)^2 + \cdots\right] - 1}.\end{aligned}$$

In the linear approximation of the Taylor series expansion,

$$\rho_\lambda(\lambda, T) = \frac{8\pi hc}{\lambda^5}\,\frac{\lambda k_{\mathrm{B}}T}{hc} = \frac{8\pi}{\lambda^4}\,k_{\mathrm{B}}T,$$

i.e., Rayleigh–Jeans formula [Eq. (1.5)].

The condition $hc/\lambda k_{\mathrm{B}}T \ll 1$ is equivalent to $hc/\lambda \ll k_{\mathrm{B}}T$, i.e., the quantum of energy hc/λ is much smaller than the thermal energy. The discreteness of cavity energy is unsignificant and the treatment of thermal radiation is well approximated by the classical theory.

(b) For $hc/\lambda k_{\mathrm{B}}T \gg 1$ we have $\exp(hc/\lambda k_{\mathrm{B}}T) \gg 1$ and

$$\rho_\lambda(\lambda, T) \approx \frac{8\pi hc}{\lambda^5}\,\frac{1}{\exp(hc/\lambda k_{\mathrm{B}}T)} = \frac{8\pi hc}{\lambda^5}\exp(-hc/\lambda k_{\mathrm{B}}T).$$

In this case $hc/\lambda \gg k_{\mathrm{B}}T$ and the discreteness of the cavity energy is obvious; the classical theory gives completely wrong predictions.

1.8 A zinc plate is irradiated at a distance $R = 1\,\mathrm{m}$ from a mercury lamp that emits through a spectral filter $P = 1\,\mathrm{W}$ radiation power at $\lambda = 250\,\mathrm{nm}$. The penetration depth of the radiation is approximately equal to the radiation wavelength. In the classical model of radiation-matter interaction, the radiation energy is equally shared by all free electrons. Calculate the minimum irradiation time for an electron to accumulate sufficient energy to escape from the metal. Free electron density in zinc is $n = 10^{29}\,\mathrm{m}^{-3}$ and the work function is $\Phi = 4\,\mathrm{eV}$.

Solution. In a time interval t the plate of area A receives the energy

$$\frac{A}{4\pi R^2}Pt$$

and this is accumulated by the free electrons in the volume $A\lambda$. On the condition that an electron acquires the energy Φ,

$$\frac{A}{4\pi R^2}Pt = nA\lambda\Phi.$$

The minimum irradiation time is

$$t = \frac{4\pi R^2 n\lambda\Phi}{P} \approx 2.0 \times 10^5\,\mathrm{s}$$

in strong contradiction to the experimental results.

1.9 When a metal surface is irradiated with light of different wavelengths from a mercury lamp, the stopping voltages of the photoelectrons are measured as shown in the following table.

λ/nm	365	405	436	546	579
U/V	1.41	1.09	0.85	0.29	0.15

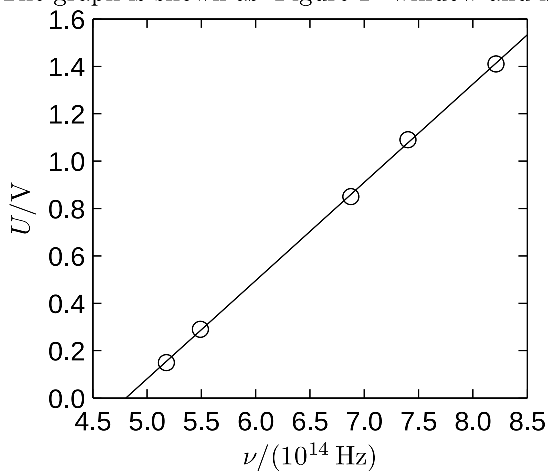
Plot the stopping voltage versus the frequency of the light and use the graph to determine the threshold frequency, the threshold wavelength, the work function of the metal, and the quotient h/e.

Solution. The dependence of the stopping voltage on the frequency of light is given by Eq. (1.10). The following MATLAB program determines all required physical quantities.

```
clc;close all;clear all
%Photoelectric effect: the use of experimental data for
%determination of nu_0, lambda_0, Phi, and h/e
c=299792458; % m/s
lambda=[365 405 436 546 579]*1e-9; % m
U=[1.41 1.09 0.85 0.29 0.15]; % V
nu=c./lambda; % Hz
nu_significand=nu*1e-14;
plot(nu_significand,U,'o');hold on
xlabel('$\nu/(10^{14} \rm{Hz})$','Interpreter','latex')
ylabel('$U/\rm{V}$','Interpreter','latex')
p=polyfit(nu_significand,U,1);
nu_0=-p(2)/p(1)*1e14 % Hz
lambda_0=c/nu_0 % m
```

```
Phi_over_e=-p(2) % V
h_over_e=p(1)*1e-14 % Vs
plot([nu_0 1.05*max(nu)]*1e-14,...
      polyval(p,[nu_0 1.05*max(nu)]*1e-14))
set(gca,'XLim',[4.5 8.5],'XTick',4.5:0.5:8.5,'XTickLabel',...
    ['4.5';'5.0';'5.5';'6.0';'6.5';'7.0';'7.5';'8.0';'8.5'],...
    'YLim',[0 1.6],'YTick',0:0.2:1.6,'YTickLabel',...
    ['0.0';'0.2';'0.4';'0.6';'0.8';'1.0';'1.2';'1.4';'1.6'])
```

After running the program, the sought quantities are displayed in the MATLAB `Command Window`: $\nu_0 \approx 4.81 \times 10^{14}$ Hz, $\lambda_0 \approx 624$ nm, $\Phi \approx 1.99$ eV, and $h/e \approx 4.15 \times 10^{-15}$ V s. The graph is shown as 'Figure 1' window and is presented below.



1.10 Blue light of wavelength $\lambda = 456$ nm and power $P = 1$ mW is incident on a photosensitive surface of cesium. The electron work function of cesium is $\Phi = 1.95$ eV.

(a) Determine the maximum velocity of the emitted electrons and the stopping voltage.

(b) If the quantum efficiency of the surface is $\eta = 0.5\,\%$, determine the magnitude of the photocurrent. The quantum efficiency is defined as the ratio of the number of photoelectrons to that of incident photons.

Solution. (a) The threshold wavelength of the photoelectric effect for cesium is

$$\lambda_0 = hc/\Phi \approx 636\,\text{nm} < \lambda,$$

so electrons are extracted from cesium.

To calculate the maximum velocity of the photoelectrons, we make use of Eq. (1.9), where $\nu = c/\lambda$. We get $v_{\max} \approx 5.2 \times 10^5\,\text{m s}^{-1}$. As $v_{\max}/c \ll 1$, the nonrelativistic expression of the kinetic energy proves to be justified.

The stopping voltage is given by Eq. (1.10), where $\nu = c/\lambda$. We get $U_0 \approx 0.77\,\mathrm{V}$.
(b) The number of electrons extracted in unit time is

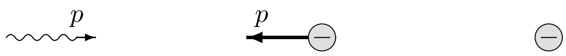
$$n = \eta \frac{P}{hc/\lambda}$$

forming a current of magnitude

$$I = ne = \eta \frac{\lambda e P}{hc} \approx 1.8\,\mu\mathrm{A}.$$

1.11 Prove that a free electron cannot absorb a photon.

Solution. The collision between the photon and the electron is investigated in their centre of mass reference frame, i.e., the frame of reference in which the total momentum is zero. The hypothetical process is presented below. Here, p denote the magnitude of the momentum of the electron and photon.



Before absorption After absorption

Let us denote by m_e the electron mass. The initial energy of the system is $pc + \sqrt{m_\mathrm{e}^2 c^4 + p^2 c^2}$, while the energy of the final state is $m_\mathrm{e} c^2$. It is clear that conservation of energy is violated, so the process cannot occur.

Remark. An electron participating in the photoelectric effect is not free, but bound to either an atom, molecule, or a solid. The electron and the heavy matter to which the electron is coupled share the energy and momentum absorbed and it is always possible to satisfy both momentum and energy conservation. However, this heavy matter carries only a very small fraction of the photon energy, so that it is usually not considered at all.

1.12 In a television tube, electrons are accelerated by a potential difference of 25 kV. Determine the minimum wavelength of the x-rays produced when the electrons are stopped at the screen.

Solution. The energy acquired by an electron accelerated by the potential difference $U = 25\,\mathrm{kV}$ is $E = eU$. This energy may be radiated as a result of electron stopping; the minimum wavelength radiated is obtained when all energy is radiated as a single photon:

$$\lambda_{\mathrm{min}} = hc/E = hc/eU \approx 5.0 \times 10^{-11}\,\mathrm{m} = 50\,\mathrm{pm}.$$

Almost all of this radiation is blocked by the thick leaded glass in the screen.

1.13 X-rays of wavelength 70.7 pm are scattered from a graphite block.

(a) Determine the energy of a photon.

(b) Determine the shift in the wavelength for radiation leaving the block at an angle of 90° from the direction of the incident beam.

(c) Determine the direction of maximum shift in wavelength and the magnitude of this shift.

(d) Determine the maximum shift in the wavelength for radiation scattered by an electron tightly bound to its carbon atom.

Solution. (a) $E = hc/\lambda \approx 2.81 \times 10^{-15}\,\text{J} \approx 1.75 \times 10^4\,\text{eV}$.

Remark. This energy is several orders of magnitude larger than the binding energy of the outer carbon electrons, so treating these electrons as free particles in the Compton effect is a good approximation.

(b) By making use of Eq. (1.12), we get $\Delta\lambda = \lambda' - \lambda \approx 2.43\,\text{pm}$.

(c) The direction of maximum shift in wavelength is $\theta = \pi$, i.e., the photon is scattered backwards; $(\Delta\lambda)_{\text{max}} = 2\lambda_{\text{C}} \approx 4.85\,\text{pm}$.

(d) The photon collides with the entire atom whose mass is 12 u. Compton wavelength of the carbon atom is $1.11\times10^{-16}\,\text{m}$. The maximum change of the wavelength due to scattering is $2.22\times10^{-16}\,\text{m}$, too small to be measured.

1.14 In a Compton scattering experiment (see Fig. 1.1) a photon of energy E is scattered by a stationary electron through an angle θ.

(a) Determine the angle φ between the direction of the recoiling electron and that of the incident photon.

(b) Determine the kinetic energy of the recoiling electron.

Solution. (a) Given that $\boldsymbol{p}$ is the momentum of the incident photon and $\boldsymbol{p}'$ and $\boldsymbol{p}_{\text{e}}$ the momenta of scattered photon and electron after the collision, the conservation of momentum requires that

$$\boldsymbol{p}_{\text{e}} = \boldsymbol{p} - \boldsymbol{p}'.$$

This relation is projected on two perpendicular axes shown in the figure:

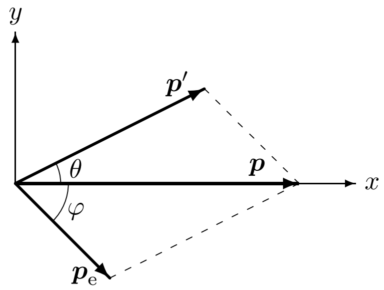
$$p_{\text{e}} \cos\varphi = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta,$$
$$p_{\text{e}} \sin\varphi = \frac{h}{\lambda'}\sin\theta.$$

Dividing side by side the two equations we get

$$\cot\varphi = \frac{\lambda'}{\lambda\sin\theta} - \cot\theta.$$

By use of Eq. (1.12), the angle φ is given by

$$\cot\varphi = \left(1 + \frac{\lambda_{\text{C}}}{\lambda}\right)\tan\frac{\theta}{2} = \left(1 + \frac{E}{mc^2}\right)\tan\frac{\theta}{2}.$$



Momentum conservation in the Compton effect and the choice of a xOy coordinate system.

(b) The kinetic energy of the recoiling electron is equal to the energy loss of the photon:

$$T = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{hc}{\lambda}\,\frac{\lambda_C(1-\cos\theta)}{\lambda+\lambda_C(1-\cos\theta)} = \frac{(E/mc^2)(1-\cos\theta)}{1+(E/mc^2)(1-\cos\theta)}E.$$

1.15 An atom in an excited state spontaneously de-excite to a lower energy state. Let us denote by A the Einstein coefficient for the spontaneous emission betweend to states. Justify the name of *lifetime of the excited state* for the time interval defined by $\tau = 1/A$.

Solution. We calculate the average time spent by the atom before de-excitation. Suppose at $t = 0$ there are $N(0)$ atoms in the upper energy level. During the time interval $(t - \mathrm{d}t/2, t + \mathrm{d}t/2)$, a number

$$AN(t)\,\mathrm{d}t = AN(0)\exp(-At)\,\mathrm{d}t$$

of atoms de-excite. These atoms have spent a time t in the upper energy level. Thus, the probability that an atom remains in the upper energy level a time t is

$$\frac{AN(t)\,\mathrm{d}t}{N(0)} = A\exp(-At)\,\mathrm{d}t.$$

The average time spent by an atom in the upper energy level is

$$\int_0^\infty tA\exp(-At)\,\mathrm{d}t = -t\exp(-At)\big|_0^\infty + \int_0^\infty \exp(-At)\,\mathrm{d}t = \frac{1}{A} = \tau.$$

Appendix A

Fundamental physical constants

The tables below give values of some basic physical constants recommended for international use by the Committee on Data for Science and Technology (CODATA).

Table A.1 An abbreviated list of the 2014 CODATA recommended values of the fundamental constants of physics and chemistry. The standard uncertainty in the last two digits is given in parenthesis.

Quantity	Symbol	Value
speed of light in vacuum	c, c_0	$299\,792\,458\,\mathrm{m\,s^{-1}}$ (exact)
magnetic constant	μ_0	$4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}}$
electric constant $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,817... \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Newtonian constant of gravitation	G	$6.674\,08(31) \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Avogadro constant	N_A	$6.022\,140\,857(74) \times 10^{23}\,\mathrm{mol^{-1}}$
molar gas constant	R	$8.314\,4598(48)\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Boltzmann constant R/N_A	k	$1.380\,648\,52(79) \times 10^{-23}\,\mathrm{J\,K^{-1}}$
		$8.617\,3303(50) \times 10^{-5}\,\mathrm{eV\,K^{-1}}$
molar volume of ideal gas		
($T = 273.15\,\mathrm{K}$, $p = 101.325\,\mathrm{kPa}$)	V_m	$22.413\,962(13) \times 10^{-3}\,\mathrm{m^3\,mol^{-1}}$
Loschmidt constant		
($T = 273.15\,\mathrm{K}$, $p = 101.325\,\mathrm{kPa}$)	n_0	$2.686\,7811(15) \times 10^{25}\,\mathrm{m^{-3}}$
elementary charge	e	$1.602\,176\,6208(98) \times 10^{-19}\,\mathrm{C}$
Faraday constant $N_\mathrm{A}e$	F	$96\,485.332\,89(59)\,\mathrm{C\,mol^{-1}}$
Planck constant	h	$6.626\,070\,040(81) \times 10^{-34}\,\mathrm{J\,s}$
		$4.135\,667\,662(25) \times 10^{-15}\,\mathrm{eV\,s}$

Table A.1 (*Continued*)

Quantity	Symbol	Value
$h/2\pi$	$\hbar$	$1.054\,571\,800(13) \times 10^{-34}$ J s
		$6.582\,119\,514(40) \times 10^{-16}$ eV s
electron mass	m_{e}	$9.109\,383\,56(11) \times 10^{-31}$ kg
energy equivalent in MeV	$m_{\mathrm{e}}c^2$	$0.510\,998\,9461(31)$ MeV
electron charge to mass quotient	$-e/m_{\mathrm{e}}$	$-1.758\,820\,024(11) \times 10^{11}$ C kg^{-1}
proton mass	m_{p}	$1.672\,621\,898(21) \times 10^{-27}$ kg
$m_{\mathrm{p}} = A_{\mathrm{r}}(\mathrm{p})$ u		$1.007\,276\,466\,879(91)$ u
energy equivalent in MeV	$m_{\mathrm{p}}c^2$	$938.272\,0813(58)$ MeV
neutron mass	m_{n}	$1.674\,927\,471(21) \times 10^{-27}$ kg
$m_{\mathrm{n}} = A_{\mathrm{r}}(\mathrm{n})$ u		$1.008\,664\,915\,88(49)$ u
energy equivalent in MeV	$m_{\mathrm{n}}c^2$	$939.565\,4133(58)$ MeV
proton-electron mass ratio	$m_{\mathrm{p}}/m_{\mathrm{e}}$	$1836.152\,673\,89(17)$
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5664(17) \times 10^{-3}$
inverse fine-structure constant	$1/\alpha$	$137.035\,999\,139(31)$
Rydberg constant $\alpha^2 m_{\mathrm{e}}c/2h$	R_∞	$10\,973\,731.568\,508(65)$ m^{-1}
$R_\infty hc$ in eV		$13.605\,693\,009(84)$ eV
Wien displacement law constants		
$b = \lambda_{\mathrm{max}}T$	b	$2.897\,7729(17) \times 10^{-3}$ m K
$b' = \nu_{\mathrm{max}}/T$	b'	$5.878\,9238(34) \times 10^{10}$ Hz K^{-1}
Stefan–Boltzmann constant		
$(\pi^2/60)k^4/\hbar^3c^2$	σ	$5.670\,367(13) \times 10^{-8}$ W m^{-2} K^{-4}
Bohr radius $\alpha/4\pi R_\infty = 4\pi\epsilon_0\hbar^2/m_{\mathrm{e}}e^2$	a_0	$0.529\,177\,210\,67(12) \times 10^{-10}$ m
Compton wavelength $h/m_{\mathrm{e}}c$	λ_{C}	$2.426\,310\,2367(11) \times 10^{-12}$ m
classical electron radius $\alpha^2 a_0$	r_{e}	$2.817\,940\,3227(19) \times 10^{-15}$ m
Bohr magneton $e\hbar/2m_{\mathrm{e}}$	μ_{B}	$927.400\,9994(57) \times 10^{-26}$ J T^{-1}
		$5.788\,381\,8012(26) \times 10^{-5}$ eV T^{-1}
nuclear magneton $e\hbar/2m_{\mathrm{p}}$	μ_{N}	$5.050\,783\,699(31) \times 10^{-27}$ J T^{-1}
		$3.152\,451\,2550(15) \times 10^{-8}$ eV T^{-1}

Table A.2 The values in SI units of some non-SI units.

Name of unit	Symbol	Value in SI units
ångström	Å	$0.1\,\mathrm{nm} = 100\,\mathrm{pm} = 10^{-10}\,\mathrm{m}$
electron volt[a]: (e/C) J	eV	$1.602\,176\,6208(98) \times 10^{-19}\,\mathrm{J}$
(unified) atomic mass constant[b] $1\,\mathrm{u} = m_\mathrm{u}$ $= (1/12)m(^{12}\mathrm{C}) = 10^{-3}\,\mathrm{kg\,mol^{-1}}/N_\mathrm{A}$	u	$1.660\,539\,040(20) \times 10^{-27}\,\mathrm{kg}$

[a]The electronvolt is the kinetic energy acquired by an electron in passing through a potential difference of one volt in vacuum.

[b]The unified atomic mass constant is equal to 1/12 times the mass of a free carbon 12 atom, at rest and in its ground state.