1. Equation of electromagnetic waves in vacuum

In vacuum there are no electric charges $\rho = 0$ and no electric currents $\vec{j} = 0$. Maxwell equations in differential form (MxI'-MxIV') become:

$$\nabla \vec{E} = 0 \qquad \nabla \vec{B} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(EM36)}$$

Apply $\nabla \times$ to the 3rd equation (EM36) and use $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\vec{c}$, with $\vec{a} \equiv \vec{b} \equiv \nabla$:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla)\vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

But $\nabla \vec{E} = 0$ and $\nabla \cdot \nabla = \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The product $\varepsilon_0 \mu_0$ is the inverse of the square of the light velocity in vacuum; eventually we find the3D wave equation:

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \Delta \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \text{(EM37)}$$

The same equation holds for every field or induction. In certain particular conditions all the conclusions find for elastic waves apply to em waves as well. The d'Alembert and Fourier type solutions are correct for each component of electric or magnetic fields. Interference and diffraction are phenomena we come across in the em case as well as for elastic waves.

In the matter the above equation writes:

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{c_n^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \text{with}$$

$$c_n = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\frac{1}{\sqrt{\varepsilon_r\mu_r}} = \frac{c}{n} \qquad \text{(EM38)}$$

c_n the velocity of light in the substance, n is the refractive index

$$n = \sqrt{\varepsilon_r \mu_r} \approx \sqrt{\varepsilon_r} \qquad \text{(EM39)}$$

The last approximation holds in non-magnetic materials, where $\mu_r \approx 1$. It turns out that in the vast majority of situations $n>1$, so that $c_n<1$.

2. Characteristics of electromagnetic waves

1. *Types of solutions*

Harmonic plane waves. All the solutions given by d'Alembert are correct for each component of the electric or magnetic fields. Any wave may be developed in Fourier series or integral. Therefore we study harmonic plane waves (hpw), find some general features and using superposition we expand them to more general waves. A scalar hpw is given by:

$$\psi(\vec{r},\, t) = a \exp[i(\omega t - \vec{k}\vec{r})], \qquad \text{(EM40)}$$

or an equivalent trigonometric function. Functions may contain a phase shift or an initial phase. Quantities a, ω, $\vec{k}$ are constant. The direction of the wave vector $\vec{k}$ gives the direction of propagation. The relation between the frequency and $\vec{k}$ is:

$$\omega = \frac{c}{|\vec{k}|} \quad \text{in vacuum and} \quad \omega = \frac{c_n}{|\vec{k}|} \quad \text{in matter} \qquad \text{(EM41)}$$

Spherical waves. A point-like source would emit a spherical wave. Such a wave fades as it travels away from the source. It has the expression:

$$\psi_{sph}(\vec{r},\, t) = \frac{a \exp[i(\omega t - \vec{k}\vec{r})]}{r} \qquad \text{(EM41)}$$

For vector-type waves relations of the type (EM40-41) are correct for each component and we write e.g.

$$\vec{E}(\vec{r},\, t) = \vec{E}_0 \exp[i(\omega t - \vec{k}\vec{r})] \qquad \text{(EM42)}$$

Remark: in some handbooks the sign of the phase could be opposite. The present convention is the same as in quantum mechanics.

2. *Transversality*

Assume we are in vacuum (or in a material with no charges and currents, e.g. a good insulator). The local Maxwell equations are:

$$\mathrm{div}\vec{E} = \nabla\vec{E} = 0;\ \ \mathrm{div}\vec{B} = \nabla\vec{B} = 0;\ \ \mathrm{curl}\vec{E} = \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t};\ \ \mathrm{curl}\vec{B} = \nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \quad \text{(EM43)}$$

It's easy to show that for a hpw the operators $\frac{\partial}{\partial t}$ and ∇ are replaced by multiplications:

$$\frac{\partial}{\partial t} \to i\omega \qquad \nabla\cdot \to -i\vec{k}\,\cdot \qquad \nabla\times \to -i\vec{k}\,\times \qquad \text{(EM44)}$$

The first two eqs from (EM43) become:

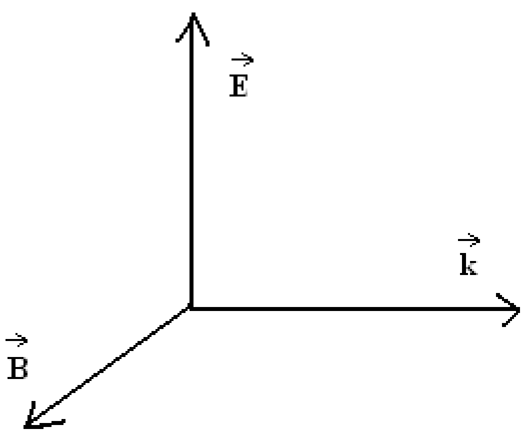
$$\vec{k}\cdot\vec{E} = 0 \qquad \vec{k}\cdot\vec{B} = 0 \qquad \text{(EM45)}$$

Electromagnetic waves are transverse.

3. *Relations between* $\vec{E}$ *and* $\vec{B}$

With (EM44) in the last two relations from (EM43):

$$\vec{B} = \frac{1}{c}\vec{u}_{\vec{k}}\times\vec{E} \qquad \vec{E} = -c\vec{u}_{\vec{k}}\times\vec{B} \qquad \text{(EM46)}$$



For fields we write

$$\sqrt{\varepsilon_0}\left|\vec{E}\right| = \sqrt{\mu_0}\left|\vec{H}\right| \qquad \text{(EM47)}$$

The expression

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 376\Omega \qquad \text{(EM48)}$$

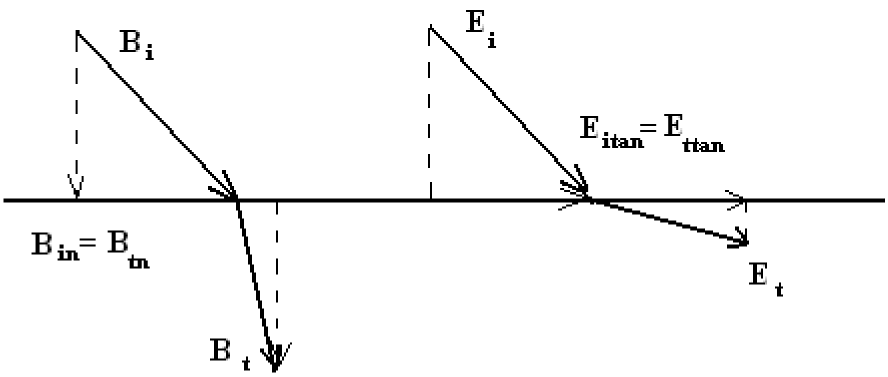
is the intrinsic impedance of the vacuum.

4. *The Helmholtz equation*. It is the equation fulfilled by a harmonic solution of the form (EM42): $\vec{E}(\vec{r}, t) = \vec{E}_0(\vec{r})\exp^{i\omega t}$. Bring it in the wave equation to find

$$\Delta\vec{E}_0(\vec{r}) + \left(\frac{\omega}{c}\right)^2 \vec{E}_0(\vec{r}) = \Delta\vec{E}_0(\vec{r}) + \vec{k}^2\vec{E}_0(\vec{r}) = \left(\nabla^2 + \vec{k}^2\right)\vec{E}_0(\vec{r}) = 0 \qquad \text{(EM49)}$$

3. Reflection and refraction

1. *Continuity conditions at discontinuity surfaces*. For surfaces without charges or currents: the normal components of inductions and the tangential components of fields are continuous, see the next figure:

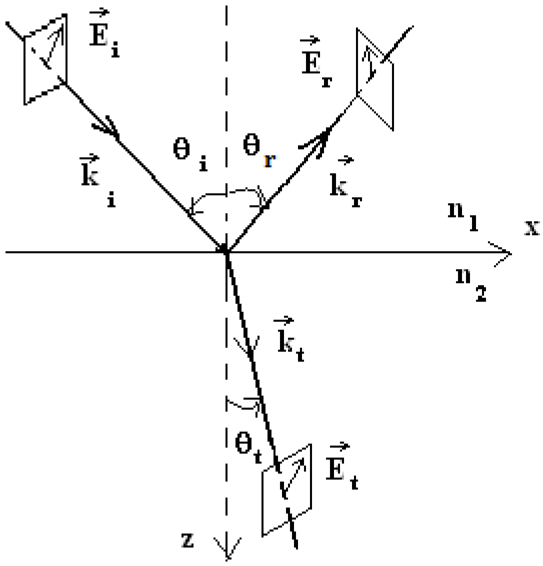


$$\vec{B}_{1n} = \vec{B}_{2n}; \qquad \vec{D}_{1n} = \vec{D}_{2n}; \qquad \vec{E}_{1\tan} = \vec{E}_{2\tan}; \qquad \vec{H}_{1\tan} = \vec{H}_{2\tan} \qquad \text{(EM50)}$$

2. *Laws of* r*eflection and refraction* embrace the phenomena which occur at the surface of separation between two materials (remarks about diffraction). One assumes the surface is (locally) plane and the incident wave is hp. The geometry is shown below. The three waves, incident, reflected and transmitted, have the form:

$$\vec{E}_i = \vec{E}_{0i}\exp\left[i\left(\omega t - \vec{k}_i\vec{r}\right)\right] \quad \vec{E}_r = \vec{E}_{0r}\exp\left[i\left(\omega' t - \vec{k}_r\vec{r}\right)\right] \quad \vec{E}_t = \vec{E}_{0t}\exp\left[i\left(\omega'' t - \vec{k}_t\vec{r}\right)\right]$$

where: $k_i = \frac{\omega}{c_{n1}} = \frac{n_1\omega}{c}$, $\quad k_r = \frac{\omega'}{c_{n1}} = \frac{n_1\omega'}{c}$, $\quad k_t = \frac{\omega''}{c_{n2}} = \frac{n_2\omega''}{c}$



Then the above continuity equations give on the surface z=0:

$$\vec{E}_{0i} \exp\left[i\left(\omega t - \vec{k}_i \vec{r}\right)\right] + \vec{E}_{0r} \exp\left[i\left(\omega' t - \vec{k}_r \vec{r}\right)\right] = \vec{E}_{0t} \exp\left[i\left(\omega'' t - \vec{k}_t \vec{r}\right)\right], \qquad z = 0$$

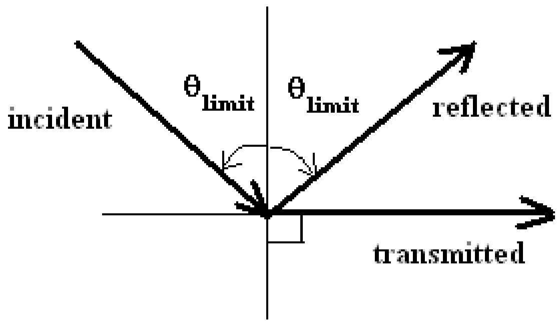
These equalities are true for each x, y and t, hence:

$$\omega = \omega' = \omega'' \qquad \theta_r = \theta_i \qquad n_1 \sin\theta_i = n_2 \sin\theta_t \qquad \text{(EM51)}$$

Let’s write all the “laws” of reflection and refraction:

1. The reflection and the refraction occur in the incident plane defined by the incidence direction and the normal to the surface.
2. The frequency remains unchanged.
3. The reflection angle equals the incident angle.
4. At refraction the Snellius-Descartes law is satisfied: $n_1 \sin\theta_i = n_2 \sin\theta_t$.

If $n_1 < n_2$ then $\theta_i > \theta_t$, the 2[nd] material is *optically more dense*. In the opposite situation $\theta_i < \theta_t$ and for each pair of materials we find an incident angle for which the refracted wave moves along the surface; this is the *limit angle*.



This is called **total reflection**.

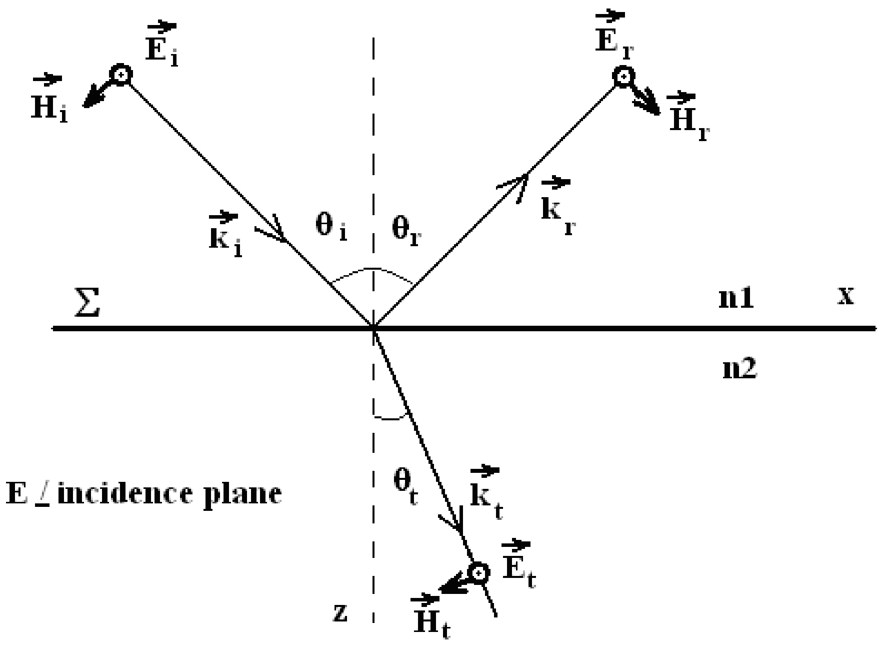
The condition is

$$\sin\theta_l = \frac{n_2}{n_1} \qquad \text{(EM52)}$$

and can be fulfilled only if $n_2 < n_1$. This phenomenon is the foundation of all the devices based on optical fibers and guides.

3. *The Fresnel relations*.

Geometry for the electric field normal to the incident plane:



If $\vec{\mathbf{E}}_i$ // incident plane, the magnetic field is normal to this plane. The figure can be deduced from the one above, keeping in mind that $\vec{\mathbf{E}} \times \vec{\mathbf{H}} \,//\, \vec{\mathbf{k}}$.

$\vec{\mathbf{E}}_i = E_i \vec{\mathbf{u}}_y \qquad \vec{\mathbf{H}}_i = \frac{1}{Z_i} E_i \vec{\mathbf{u}}_i \times \vec{\mathbf{u}}_y$

where: $\vec{\mathbf{u}}_i = \sin\theta_i \, \vec{\mathbf{u}}_x + \cos\theta_i \, \vec{\mathbf{u}}_z$

The continuity of the tangential components of the electric field writes as:

$$E_{0i} + E_{0r} = E_{0t}$$

and that of the magnetic field is:

$$(H_{0r} - H_{0i})\cos\theta_i = -H_{0t}\cos\theta_t$$

or:

$$\left(E_{0i} - E_{0r}\right)\frac{\cos\theta_i}{Z_1} = E_{0t}\,\frac{\cos\theta_t}{Z_2}$$

Ratios:

$$r_{perp} = \left(\frac{E_{0r}}{E_{0i}}\right)_{perp} \qquad t_{perp} = \left(\frac{E_{0t}}{E_{0i}}\right)_{perp}$$

are known as the Fresnel coefficients for reflection and transmission in the perpendicular case. They are obtained immediately from the above relations:

$$r_{perp} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = -\frac{\sin\left(\theta_i - \theta_t\right)}{\sin\left(\theta_i + \theta_t\right)} \qquad \text{(EM52)}$$

$$t_{perp} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{2\cos\theta_i\sin\theta_t}{\sin\left(\theta_i + \theta_t\right)} \qquad \text{(EM53)}$$

The equalities are exact in non-magnetic materials, where $\frac{Z_1}{Z_2} = \frac{n_2}{n_1}$.

For the parallel situation, $\vec{\mathbf{E}}_i$ // incident plane, the results are:

$$r_{//} = \left(\frac{E_{0r}}{E_{0i}}\right)_{//} \qquad t_{//} = \left(\frac{E_{0t}}{E_{0i}}\right)_{//}$$

$$r_{//} = \frac{Z_1 \cos\theta_i - Z_2 \cos\theta_t}{Z_1 \cos\theta_i + Z_2 \cos\theta_t} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \qquad \text{(EM54)}$$

$$t_{//} = \frac{2Z_2 \cos\theta_i}{Z_1 \cos\theta_i + Z_2 \cos\theta_t} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \qquad \text{(EM55)}$$

Exercise 1. Show that:

$$\frac{n_2}{n_1} t_{//} - r_{//} = 1 \qquad\qquad t_{perp} - r_{perp} = 1$$

Energy transfer by reflection and transmission.

The reflection factor is defined by:

$$R = \frac{\text{flux of reflected energy}}{\text{flux of incident energy}}$$

The transmission factor is defined by:

$$T = \frac{\text{flux of transmitted energy}}{\text{flux of incident energy}}$$

The results are:

$$R = \frac{n_1}{n_1} \frac{\vec{\mathbf{u}}_r \cdot \vec{\mathbf{u}}_n}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_n} \frac{E_{0r}^2}{E_{0i}^2} = r^2 \qquad \text{(EM56)}$$

$$T = \frac{n_2}{n_1} \frac{\vec{\mathbf{u}}_t \cdot \vec{\mathbf{u}}_n}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_n} \frac{E_{0t}^2}{E_{0i}^2} = \frac{n_2 \cos\theta_t}{n_1 \cos\theta_i} t^2 \qquad \text{(EM57)}$$

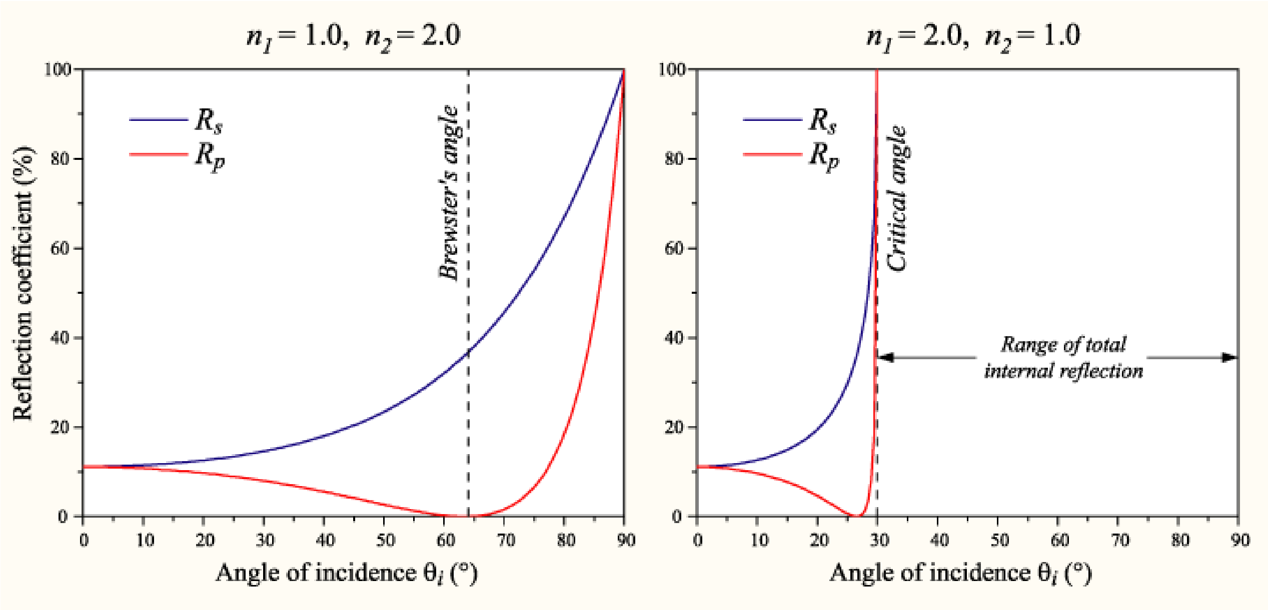
As we have neglected absorption, $T+R=1$.

From (EM54) we see that if $\tan(\theta_i + \theta_t) \to \infty$, i.e. if $\theta_i + \theta_t = \pi/2$, there is no reflected wave with electric field parallel to the incidence plane. The corresponding incident angle is *the Brewster angle* θ_B. The condition is

$$\theta_i + \theta_t = \pi/2, \qquad\qquad \tan\theta_B = \frac{n_2}{n_1} \qquad \text{(EM58)}$$

For this angle of incidence, $r_{//}(\theta_B) = 0$ and the reflected light is totally polarized with the electric field perpendicular to the incidence plane.

From Wikipedia, the article *The Fresnel equations*



For application to optical fibers see

http://en.wikipedia.org/wiki/Optical_fiber