Relativistic dynamics

We begin with the four-momentum

$$\mathcal{P} = m_0 \mathcal{V} = \gamma\left(m_0\vec{v};\ i m_0 c\right) \tag{12}$$

The spatial part may be named *relativistic momentum*:

$$\vec{p} = \gamma m_0 \vec{v} = \frac{m_0\vec{v}}{\sqrt{1-\beta^2}} = m\vec{v} \tag{13}$$

if we denote by *m* the *relativistic mass*:

$$m = \frac{m_0}{\sqrt{1-\beta^2}} > m_0 \tag{14}$$

This mass depends on the speed of the body and becomes infinity when $v \to c$. This gives an explanation why *all bodies have velocities which do not exceed the speed of light in vacuum. The only entities that move with the speed of light must have zero rest mass.* (why?)

What is the meaning of the temporal part of $\mathcal{P}$? To work out this meaning, we begin with the 2nd law written in the original form given by Newton: $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$. Choose the axis along the force and compute work:

$$W = \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} \frac{d}{dt}(mv)dx = \int_{x_1}^{x_2} d(mv)\frac{dx}{dt} = \int_{x_1}^{x_2} v d(mv) = mv^2\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} mvdv$$

Use now (14) to transform the last integral:

$$\int_{x_1}^{x_2} mvdv = m_0\int_{x_1}^{x_2} \frac{vdv}{\sqrt{1-v^2/c^2}} = -m_0c^2\sqrt{1-v^2/c^2}\Big|_{x_1}^{x_2} = -mc^2\left(1-v^2/c^2\right)\Big|_{x_1}^{x_2} =$$

$$-m(v_2)c^2 + m(v_2)v_2^{\,2} + m(v_1)c^2 - m(v_1)v_1^{\,2}$$

$$W = \left(m(v_2) - m(v_1)\right)c^2 \tag{15}$$

But $W = E_2 - E_1$. Therefore we interpret the quantity mc^2 as the total energy:

$$\boxed{E = m(v)c^2} \tag{26}$$

Hence the 4-momentum becomes the 4-vector momentum-energy:

$$\mathcal{P} = \left(\vec{p};\, i\frac{mc^2}{c} \right) = \left(\vec{p};\, i\frac{E}{c} \right) \tag{27}$$

Other important forms of energy:

Rest energy, the energy measured in the proper frame:

$$E_0 = m_0 c^2 \tag{28}$$

Kinetic energy in relativity:

$$E_k = E - E_0 = (m - m_0)c^2 = m_0(\gamma - 1)c^2 = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) \tag{29}$$

This kinetic energy corresponds to the non-relativistic form only for small velocities:

$$E_k = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) \approx m_0 c^2 \left(1 + \frac{\beta^2}{2} - 1 \right) = \frac{m_0 v^2}{2}, \text{ for } \beta << 1$$

Write (16) $\mathcal{P}^2 = -\, m_0^2 c^2$ in extended form:

$$p^2 - \frac{E^2}{c^2} = -m_0^2 c^2 \quad \rightarrow \quad E^2 = p^2 c^2 + m_0^2 c^4 \tag{30}$$

Problem: Taking the square root of (30) do we need to consider both signs? Energy jumps and energy quantification.

Particles moving with the speed of light

The mass $m(v) \xrightarrow[v \to c]{} \infty$. Are there particles moving with the speed of light? Obviously yes, at least the photons. In order to keep finite quantities we have to admit that the rest mass of such particles is nil. Eq. (30) reduces to $E_{ph} = p_{ph}c$, and as $p=mv=mc$, we find again Eq. (26) for particles moving with the speed of light. The mass is defined by $m_{ph} = \frac{p_{ph}}{c}$.

Problem. Show that a free particle (with non-zero rest mass) can not absorb or emit a photon.

The absorption reaction is:

$$\text{particle + photon} \rightarrow \text{(excited) particle}$$

The initial 4-momentum is $\mathcal{P}_{\text{in}} = \mathcal{P}_{\text{part}} + \mathcal{P}_{\text{ph}}$ and the final one is $\mathcal{P}_{\text{fin}} = \mathcal{P}_{\text{part}}$. The square of a 4-vector is a relativistic invariant. Write the square of the initial four-momentum in the rest frame of the initial particle and that of the final particle in its rest frame:

$$\mathcal{P}_{\text{in}} = (0;\, im_0c) + (\vec{p}_{ph};\, ip_{ph}) \qquad \mathcal{P}_{\text{fin}} = (0;\, im_0c)$$

$$(\mathcal{P}_{\text{in}})^2 = -m_0^2c^2 - 2m_0cp_{ph} = -m_0^2c^2 - 2m_0E_{ph} \qquad (\mathcal{P}_{\text{fin}})^2 = -m_0^2c^2$$

The only possibility to conserve the square would be to put $E_{ph} = 0$, but then the photon exists no more.

Consider in a similar way the emission process.

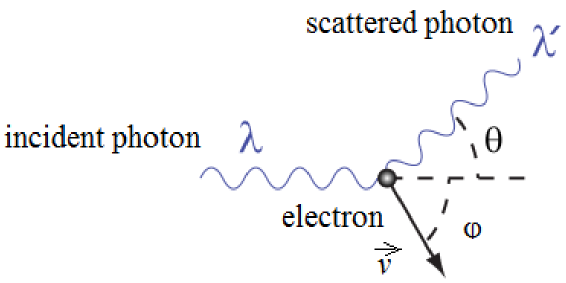
Compton effect (**http://en.wikipedia.org/wiki/Compton_scattering**)

The Compton effect is the inelastic collision between a high energy photon (*X*– or γ–ray) and an electron. It was discovered and explained in 1923.

Remark: the Compton scattering may appear also in photon-nuclear collisions, but the effect is much smaller (explain why).

Experiment: when a mono-energetic (or monochromatic) beam of photons hits a thin foil of a solid material photons are scattered at various angles θ. During these scattering processes photons loose energy. They emerge with longer wavelengths (smaller energies) which fulfill the Compton relation

$$\boxed{\lambda' - \lambda = 2\frac{h}{m_{0e}c}\sin^2\frac{\theta}{2}} \qquad \text{(Compton)}$$



The Compton relation may be proved using relativistic 4-vectors or usual momentum and energy conservation theorems. See the first metod e.g. in Wikipedia. For the second method assume the *e* initially at rest and then moving with velocity $\vec{v}$ after the interaction. Write down:

- the conservation of energy

$$h\nu + m_{0e}c^2 = h\nu' + m_e(v)c^2 \qquad (*)$$

- the momentum conservation along the horizontal direction

$$h\nu / c = h\nu' / c + m_e(v)v\cos\varphi \qquad (**)$$

- the momentum conservation along the vertical direction

$$0 = h\nu' / c\sin\theta - m_e(v)v\sin\varphi \qquad (***)$$

Eliminate φ and v from (*), (**), (***) and find (Compton).