Waves

What are waves ?

An oscillation is a periodic movement of a material point. 1D harmonic oscillations are sinusoidal movements; the same is true for 3D harmonic oscillations, but now the amplitude and the elongation are vectorial quantities. For example 1D free oscillations movement is described by the function:

$$\psi(t) = a\sin(\omega t + \varphi_0) \qquad \text{(W1)}$$

The Greek letter ψ represents the movement of a single point.

A wave is the propagation of an oscillation or of a disturbance in a medium. It appears when the local movement of a point changes the positions and the movements of adjacent material points. These changes in turn modify the positions and the movement of other points, the result being changes in many places.

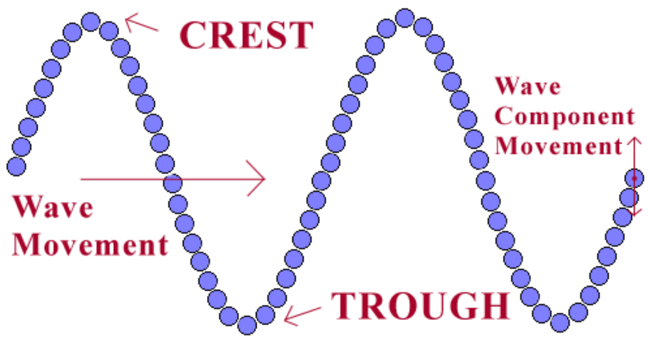
Remark: this description applies mainly to elastic waves. Electromagnetic waves are not to be associated with macroscopic movements, but with macroscopic changes of electric and magnetic fields in a whole region.

Types of waves: **longitudinal** (sound in air, compression waves in rods) and **transverse** (electromagnetic, sound in strings).

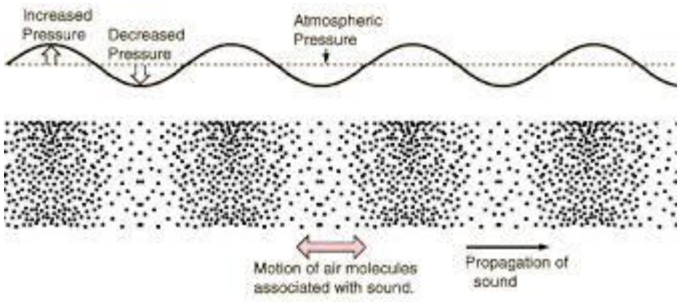
See:

https://www.google.ro/search?q=transverse+waves&rlz=1C2FDUM_enRO472RO472&tbm=isch&tbo=u&source=univ&sa=X&ei=KnqTUraLM7P2yAPhxoHoCg&ved=0CCwQsAQ&biw=1513&bih=752

for transverse and longitudinal waves



Transverse wave



Longitudinal wave – sound

Polarization of elastic waves

Notice: there is a huge difference between the velocity of the movement of a point, defined as $\dot{\psi}(t) = \frac{\mathrm{d}\psi}{\mathrm{d}t}$ and the propagation velocity of the wave v_p; the definition will be given later, but a good estimate is to consider it as the distance the wave covers in a second. Example: a wave on the sea moves towards the shore at, say, 5m/s; this is not at all the velocity with which water drops move inside the wave, mainly up and down.

Let's assume a 1D medium, like a string and locate a point along the string with abscissa x. When the time has the value t the movement of the material point situated at abscissa x is given by a function of both variables $\psi(x, t)$. Assume further that the medium is homogeneous, such that the propagation velocity v_p is a constant. If the source of oscillations is placed in the origin $x_0 = 0$ and has no phase shift, its equation is $\psi(0, t) = a\sin(\omega t)$. This is a *harmonic oscillation*, because it contains a single frequency. What would be the movement of the point situated at abscissa x? What would be the expression for $\psi(x, t)$? Obviously, what happens in the origin at the moment t=0 should happen in the point x after the *propagation time* given by $t_p = x/v_p$. So we write:

$$\psi(x, t) = a\sin[\omega(t - t_p)] = a\sin\left(\omega t - \frac{\omega x}{v_p}\right) = a\sin(\omega t - kx) \qquad \text{(W2)}$$

where

$$k = \frac{\omega}{v_p} \qquad \text{(W3)}$$

is the *wave vector*. Eq. (W2) represents a 1D wave traveling in the *Ox* direction. The same could be said about $a\,cos(\omega t - kx)$ or $a\exp[i(\omega t - kx)]$.

Question: what can we say about the expression $a\exp[i(\omega t + kx)]$?

Equation of a 1D wave

What is the equation satisfied by the function $\psi(x, t)$? Let's compute the second partial derivatives of (W2):

$$\frac{\partial^2 \psi}{\partial x^2} = k^2 a \sin(\omega t - kx) = \frac{\omega^2}{v_p^2} a \sin(\omega t - kx) \qquad \frac{\partial^2 \psi}{\partial t^2} = \omega^2 a \sin(\omega t - kx)$$

Hence, by dividing the time derivative with the constant v_p^2 we get a term equal to the spatial derivative. **The wave equation** is obviously:

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v_p^2}\frac{\partial^2 \psi}{\partial t^2} = 0} \qquad \text{(W4)}$$

In 3D the equation is:

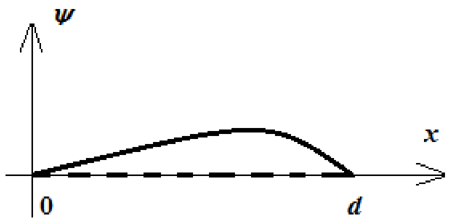
$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{v_p^2}\frac{\partial^2 \psi}{\partial t^2} = 0} \qquad \text{(W5)}$$

with $\psi(x,\ y,\ z,\ t)$ the quantity describing the type of wave: displacement, pressure, electric or magnetic field, etc.

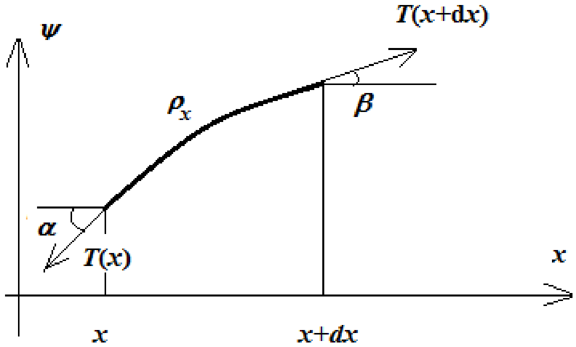
Exercise: Verify that (W2) is a solution of (W4).

Equation of 1D transverse waves in a string

Assume a homogeneous string of length *d* and constant section. Denote by ρ_x its linear density ($kg \cdot m^{-1}$). In equilibrium the string lies along the Ox axis. Out of equilibrium, its position appears as in Fig. below:



Let's take a small part of the perturbed string, between x and x+dx. Tension $\vec{T}(x)$ and $\vec{T}(x+dx)$ act on the two ends. Assume their moduli are equal, $T(x) \cong T(x+\mathrm{d}x)$.



The wave being transverse, all the movement is done along the vertical direction and Newton 2nd law for the portion (x, x+dx) writes:

$$\mathrm{d}m \cdot \frac{\partial^2 \psi}{\partial t^2} = T(x+\mathrm{d}x)\sin\beta - T(x)\sin\alpha$$

Angles are very small, sin functions may be replaced by tan functions, these being equal to derivatives with respect to x:

$$\rho_x \mathrm{d}x \cdot \frac{\partial^2 \psi}{\partial t^2} = T(x)(\tan\beta - \tan\alpha) = T(x)\left[\frac{\partial \psi(x+\mathrm{d}x)}{\partial x} - \frac{\partial \psi(x)}{\partial x}\right] = T(x)\frac{\partial^2 \psi}{\partial x^2}\mathrm{d}x$$

This equation above has the form (W4), if we agree that the velocity of transverse waves is $v_t = \sqrt{\dfrac{T}{\rho_x}}$.

D'Alembert solution (method of characteristics)

D'Alembert's idea: variable changes $u = x - v_p t$ and $w = x + v_p t$

Eq. (W4) becomes (f*or multivariable derivatives see math app 2 and 3*)

$$\frac{\partial^2 \psi}{\partial u \partial w} = 0$$

That implies the first derivative $\frac{\partial \psi}{\partial u}$ does not depend on w and vice-versa, $\frac{\partial \psi}{\partial w}$ does not depend on u. Therefore

$$\frac{\partial \psi}{\partial w} = f_2(w) \qquad\qquad \frac{\partial \psi}{\partial u} = f_1(u)$$

Multiply the first relation above with dw and the second with du and add:

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial u}\mathrm{d}u + \frac{\partial \psi}{\partial w}\mathrm{d}w = f_1(u)\mathrm{d}u + f_2(w)\mathrm{d}w$$

and we may integrate to get:

$$\psi(u, w) = f(u) + g(w)$$

or

$$\boxed{\psi(x, t) = f(x - v_p t) + g(x + v_p t)} \qquad \text{(W6)}$$

So the variables appear only as linear forms $x - v_p t$ and $x + v_p t$. f and g are **any function** that behaves nicely enough (second derivatives continuous).

Remark: our first solution (W2) does not have the form (W6) above. Where is the error? *Hint*: write $a\sin\left(\omega t - \frac{\omega x}{v_p}\right) = -a\sin\left[\frac{\omega}{v_p}(x - v_p t)\right]$ and the ratio $k = \frac{\omega}{v_p}$ is a constant.

For **Fourier solution**, see lectures. Predau metoda Fourier scriind $\psi(x,t) = X(x)\cdot T(t)$ si eparand variabilele gasesc solutia pt. $X(0)-0$, $X(a)-0$

Wave characteristics

Rewrite (W6):

$$\psi(x, t) = f(x - v_p t) + g(x + v_p t)$$

The first function is the *direct wave*, it travels towards the positive *Ox* axis.

The latter is the *inverse wave*, it travels towards the negative *Ox* axis.
Their arguments are called *phases*.
The loci of points with the same phase are known as *equiphase surfaces*. The most remote equiphase surface is the *wave front*. According to their form, waves in 3D may be plane, spherical, cylindrical, etc. For real waves these forms may be very intricate and may change during the propagation. To avoid difficulties we shall study mainly harmonic waves, with equations:

$$\psi(x, t) = a\, sin(\omega t - kx) \qquad \text{in 1D} \qquad \text{(W7)}$$

and

$$\psi(\vec{r}, t) = a \sin(\omega t - \vec{k}\vec{r}) \qquad \text{in 3D} \qquad \text{(W8)}$$

The equation of the equiphase surface for the direct wave is:

$$\omega t - kx = const \qquad \text{or} \qquad x - v_p t = const \qquad \text{(W9)}$$

Differentiating Eq. (W9):

$$\omega \mathrm{d}t - k\mathrm{d}x = 0 \qquad \text{or} \qquad \mathrm{d}x - v_p \mathrm{d}t = 0,$$

hence the *phase velocity*

$$v_p = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k} \qquad \text{(W10)}$$

This explains the meaning of the phase velocity: it is the velocity with which move the equiphase surfaces. For the inverse wave the equiphase surfaces propagate with velocity

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -v_p \qquad \text{(W10')}$$

For periodic amplitudes there are many equiphase surfaces having the same phase (mod($2m\pi$)) The distance between two adjacent equiphase surfaces is called *wavelength*.

At a certain point x_0 in space the disturbance is a function of time only: $\psi(x_0, t) = F(t)$. If this function is periodic is very important, because each frequency i the visible domain (see later electromagnetic waves) corresponds to a certain colour. Let's then consider that $F(t) = a\sin(\omega \mathrm{t} + \alpha)$, where a is called *amplitude* and the argument of the cosine function is the *phase*. The quantity

$$\nu = \frac{\omega}{2\pi} = \frac{1}{T} \qquad \text{(W11)}$$

is called *frequency* and represents the number of oscillations per second. The *angular frequency* ω is the number of vibrations per 2π seconds. T is the *period* of the vibrations. Such types of solutions are called *harmonic* with respect to time.

A very simple and important case is the *harmonic plane wave* (hpw) given by

$$\psi(x,t) = a\sin(\omega t - kx) = a\sin\left[\omega\left(\mathrm{t} - \mathrm{x}/\mathrm{v_p}\right)\right] \qquad \text{(W7')}$$

The equality $\psi(x,t) = \psi(x+\lambda, t)$, where the *wavelength* λ is defined by

$$\boxed{\lambda = v_p \frac{2\pi}{\omega} = v_p T} \qquad \text{(W12)}$$

The function $\psi(x,t)$ or $\psi(\vec{r},t)$ is also called *amplitude*. The energy transported by the wave is proportional with the square of the amplitude. The energy passing in the unit of time through the unit of surface is the *intensity* (units of W/m^2). We shall enter into details for_electromagnetic waves only.

Interference and diffraction

O simpla introducere, restul la unde elmgn

The most important characteristic of waves is their ability to interfere and to diffract. Every time a physical entity display interference or/and diffraction it is certainly a wave.

The interference is the superposition of waves in the same region. When waves from two or more sources superposes what we observe usually is an increase of the overall intensity. This is the *incoherent* superposition of waves. If the two sources are *coherent*, one observes a totally different spatial pattern formed by maxima and minima which remain stable during the observation. This is the so-called coherent or *stationary interference*. We shall deal with with this type of superposition when we study electromagnetic waves; it is the foundation of interferometry. Interferometry with elmgn waves is one of the most accurate measurement methods in physics.

The **diffraction** is the deviationfrom the straight propagation when a wave is incident upon an obstacle with wedges sharp enough, i.e. of the order of the wavelength.

Example: Two cosine waves.

Two waves superpose in the same region; the first travels the path x_1, the second one the path x_2. Assume:

- same frequency, therefore the same wave vector k
- the movements have the same direction (other situations are studied in the particular case of electromagnetic polarized waves)
- the same amplitude (for simpler formulae).

$$\psi_1(x,\ t) = a\cos(\omega t - kx_1)$$
$$\psi_2(x,\ t) = a\cos(\omega t - kx_2) \qquad \text{(W13)}$$

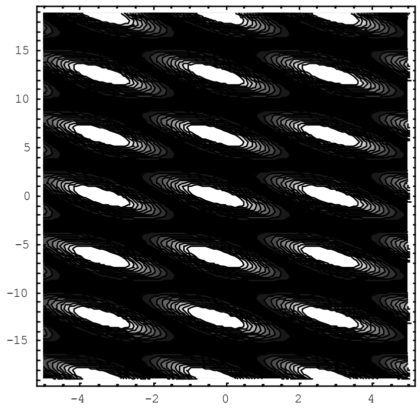
The resulting amplitude is:

$$\psi(x,\ t) = a[\cos(\omega t - kx_1) + \cos(\omega t - kx_2)] =$$
$$2a\cos\left(k\frac{x_1 - x_2}{2}\right)\cos\left(\omega t - k\frac{x_1 + x_2}{2}\right) \qquad \text{(W14)}$$

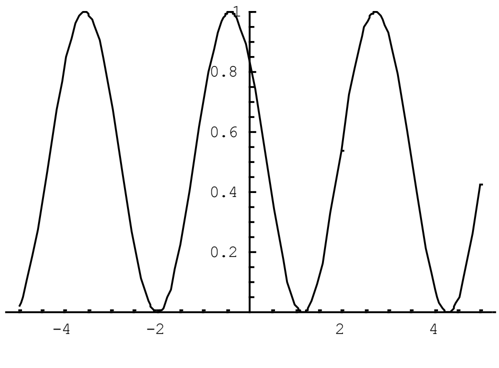
This is a cosine wave with the same frequency and with amplitude modulated by the factor $\cos\left(k\frac{x_1 - x_2}{2}\right)$. The intensity is proportional to the square of the amplitude:

$$I \propto 4a^2\cos^2\left(k\frac{x_1 - x_2}{2}\right) = 2a^2\{1 + \cos[k(x_1 - x_2)]\} = 2a^2(1 + \cos k\Delta x) \qquad \text{(W14)}$$

Here $\Delta x = x_1 - x_2$ is the difference between the paths traveled by the two waves. This is shown in the following graphs.



Intensity dependence on the horizontal position x and on the vertical phase shift $k\Delta x$. Quantity Δx is the difference in the paths of the two waves.



Intensity dependence on the horizontal position x Constant phase shift.

***Conditions for maxima and minima*:**

Max for $\quad I_{\max} = 4a^2 \quad$ for $\ k\Delta x = 2m\pi,$ or $\Delta x = m\lambda$ (W15')

Min for $I_{\min} = 0$ for $k\Delta x = (2m+1)\pi$, or $\Delta x = (2m+1)\dfrac{\lambda}{2}$ (W15'')

Question: study the case when the amplitudes are not equal.

Important Remark*:* The phase shift due to different paths is given by:

$$\Delta\varphi = k\Delta x = \frac{\omega}{v_{prop}}\Delta x = \frac{2\pi}{\lambda}\Delta x \qquad \text{(W16)}$$

In the optical domain $\lambda \cong 0.5\,\mu m$, hence a very small path difference gives a measurable phase shift. E. g. for $\Delta x = 0.005\ \mu m$, $\Delta\varphi \cong 3.6^{\circ}$, an easily detectable angle. **Optical interference is a very accurate method to measure lengths.**