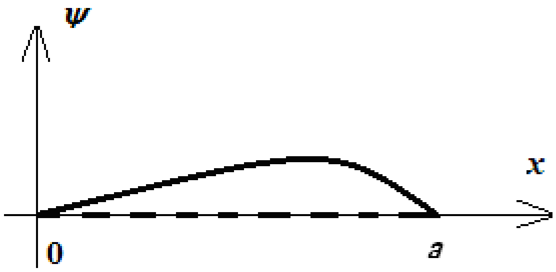
Fourier method to solve wave equation

1D waves in a string

String: the ends at x=0 and x=a are fixed, so $\psi(0,t)=0,\ \psi(a,t)=0$ (limit conditions)



Equation:

$$\boxed{\frac{\partial^2\psi}{\partial x^2}-\frac{1}{v_p^2}\frac{\partial^2\psi}{\partial t^2}=0} \qquad (1)$$

Idea: Eq. is linear (**why ?**) so a linear combination of particular solutions is a solution. (**example**)

Search a particular solution of the form

$$\psi(x,t)=X(x)T(t) \qquad (2)$$

Calculus:

$$\frac{\partial^2\psi}{\partial x^2}-\frac{1}{v_p^2}\frac{\partial^2\psi}{\partial t^2}=\frac{d^2X(x)}{dx^2}T(t)-\frac{1}{v_p^2}X(x)\frac{d^2T(t)}{dt^2}\equiv X''(x)T(t)-\frac{1}{v_p^2}X(x)T''(t)=0$$

Divide by $X(x)T(t)$ and get

$$\frac{X''(x)}{X(x)} - \frac{1}{v_p^2}\frac{T''(t)}{T(t)} = 0 \tag{3}$$

$\frac{X''(x)}{X(x)}$ depends only on *x*. $\frac{T''(t)}{T(t)}$ depends only on *t*. *Their sum is a constant (=0) only if each ratio is constant* (**discussion**) Or else the equality would be true only for certain particular values of *x* and *t*. This is not what we look for. Hence

$$\frac{X''(x)}{X(x)} = -k_x^2 = \text{const} \qquad \frac{T''(t)}{T(t)} = -\omega^2 = \text{const} \tag{4}$$

Introduce (4) in (3)

$$-k_x^2 - \frac{1}{v_p^2}\left(-\omega^2\right) = 0 \quad \Rightarrow \quad k_x^2 = \frac{\omega^2}{v_p^2} \quad \text{or} \quad k_x = \pm\frac{\omega}{v_p}. \tag{5}$$

For such a link between k_x, ω *and* v_p Eq. (3) is fulfilled, so (2) is a particular solution for (1). Eqs. (4) are simply equations for free oscillations, with solutions sin, cos or imaginary exponentials, at our convenience.

Choose

$$\psi(x,\ t) = X(x)T(t) = \left[A\sin k_x x + B\cos k_x x\right]\exp[i\omega t] \tag{6}$$

Use limit conditions

$$\psi(0,\ t) = 0 \qquad B\cos 0\exp[i\omega t] = Be^{i\omega t} = 0 \qquad \Rightarrow B = 0$$

$$\psi(a,\ t) = 0 \qquad A\sin k_x a \cdot e^{i\omega t} = 0 \qquad \Rightarrow \qquad k_x a = n\pi, \qquad n = 1, 2, 3, \ldots$$

$$k_n = \frac{n\pi}{a}, \qquad \omega_n = v_p k_n = n\frac{\pi}{a}v_p \tag{7}$$

We get an infinity of solutions depending of the index *n*:

$$\psi_n(x,\ t) = X_n(x)T_n(t) = A_n \sin\left(\frac{n\pi}{a}x\right)\exp\left[in\frac{\pi v_p}{a}t\right] \qquad (8)$$

A general solution will be

$$\psi(x,\ t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\exp\left[in\frac{\pi v_p}{a}t\right] \qquad (9)$$

Remember

$$\omega_n = n\frac{\pi}{a}v_p = \frac{2\pi}{T_n} \qquad T_n = \frac{2\pi}{\omega_n} = \frac{2a}{nv_p} \qquad (10)$$

characteristic pulsations and periods.

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{a} \qquad \lambda_n = \frac{2\pi}{k_n} = \frac{2a}{n} \qquad (11)$$

The fundamental: *n*=1, with wavelength twice the length of the string, $\lambda_1 = 2a$ and the period $T_1 = \frac{2\pi}{\omega_1} = \frac{2a}{v_p} = \frac{\lambda_1}{v_p}$.

Exercices:

1. Give v_p and *a*, find wavelengths, periods, frequencies and *k*'s for *n*=1, 2, 3,...
v_{p1}=340 m/s (sound in air), v_{p2}=3000 m/s (sound in metal); a_1=0.5 cm, *a2*=10 cm, a_3=2m,
Any other numeric application containing the above quantities.

2D waves in a membrane, 3D waves in a parallelipipedic bar

Generalization of Eq. (1) and of particular solution (2)

$$\boxed{\frac{\partial^2\psi}{\partial x^2}-\frac{1}{v_p^2}\frac{\partial^2\psi}{\partial t^2}=0} \quad \rightarrow \quad \boxed{\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{v_p^2}\frac{\partial^2\psi}{\partial t^2}=0} \qquad (1')$$

$$\psi(x,t)=X(x)T(t) \quad \rightarrow \quad \psi(x,\ y,\ z,\ t)=X(x)Y(y)Z(z)T(t) \qquad (2')$$

Limit conditions for a bar *a*x*b*x*d:*

$$\psi(0,y,\ z,\ t)=0,\ \psi(a,\ y,\ z,\ t)=0$$

$$\psi(x,0,\ z,\ t)=0,\ \psi(x,\ b,\ z,\ t)=0$$

$$\psi(x,y,\ 0,\ t)=0,\ \psi(x,\ y,\ d,\ t)=0$$

(3) becomes

$$\frac{X''(x)}{X(x)}-\frac{1}{v_p^2}\frac{T''(t)}{T(t)}=0 \rightarrow \quad \frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)}-\frac{1}{v_p^2}\frac{T''(t)}{T(t)}=0 \qquad (3')$$

(4) becomes

$$\frac{X''(x)}{X(x)}=-k_x^2 \qquad \frac{Y''(y)}{Y(y)}=-k_y^2 \qquad \frac{Z''(z)}{Z(z)}=-k_z^2 \qquad \frac{T''(t)}{T(t)}=-\omega^2 \qquad (4')$$

with the condition (5) becoming

$$k_x^2 = \frac{\omega^2}{v_p^2} \qquad \to \qquad k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{v_p^2} \tag{5'}$$

Solution as generalization of (6)

$$\psi(x,\ t) = X(x)T(t) = [A \sin k_x x + B \cos k_x x] \exp[i\omega t] \qquad \to$$

$$\psi(x, y,\ z,\ t) = [A \sin k_x x + B \cos k_x x][C \sin k_y y + D \cos k_y y][E \sin k_z z + F \cos k_z z] \exp[i\omega t]$$

Initial conditions in *x*=0, *y*=0, *z*=0 cancel cosine terms, we get

$$\psi(x, y,\ z,\ t) = A \sin k_x x \cdot C \sin k_y \ \cdot E \sin k_z z \cdot \exp[i\omega t]$$

Initial conditions in *x*=*a*, *y*=*b*, *z*=*d* generalize (7):

$$k_n = \frac{n\pi}{a}, \qquad \omega_n = v_p k_n = n \frac{\pi}{a} v_p \qquad \to$$

$$k_x = \frac{n\pi}{a} \qquad k_y = \frac{p\pi}{b} \qquad k_z = \frac{q\pi}{d}$$

$$k_x^2 + k_y^2 + k_z^2 = k_{npq}^2 = \pi^2 \left(\frac{n^2}{a^2} + \frac{p^2}{b^2} + \frac{q^2}{d^2} \right) = \frac{\omega_{npq}^2}{v_p^2} \tag{7'}$$

Exercises and numerical applications as above. Examples welcome.