Damped oscillations

A harmonic oscillator has no losses, its total energy is constant and the amplitude of the oscillations is the same in the course of the movement. This is a crude approximation. Usually there is friction. The simplest way to take friction into account is to introduce in the equation a term proportional to the velocity. Newton's 2nd law writes:

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \qquad \text{(O10)}$$

with $\gamma > 0$ a damping factor. (Why positive gammas?). The analogous of Eq. (O1) is:

$$\ddot{x} + \frac{2}{\tau}\dot{x} + {\omega_0}^2 x = 0 \qquad \text{(O11)}$$

The notation $\gamma/_m = 2/_\tau$ simplifies the final relations.

Question: What are the MU for τ ?

Solve Eq. (O11) by Euler. $r^2 + \frac{2}{\tau}r + \omega_0^2 = 0$,

$$r_{1,2} = -\frac{1}{\tau} \pm \sqrt{\left(\frac{1}{\tau}\right)^2 - \omega_0^2} \qquad \text{(O12)}$$

Three situations present:

a). $\frac{1}{\tau} > \omega_0$. **Both roots are real and both negative.** Depending on initial conditions the movement is a decay, possibly preceded by an increase.

Assume $\tau = 1s^{-1}$, $\omega_0 = 0.25s^{-1}$**.**

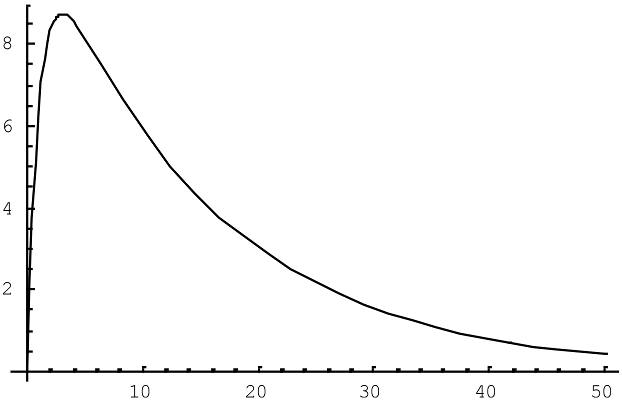
Analytic solution:

$$x(t) = e^{-t/\tau}\left(A\exp\left[\sqrt{\left(\frac{1}{\tau}\right) - \omega_0^2}\right] + B\exp\left[-\sqrt{\left(\frac{1}{\tau}\right) - \omega_0^2}\right]\right) \qquad \text{(O13)}$$

For *x*(0)=0 and $\dot{x}(0) = 1m/s$ **one finds:**

```
ClearAll[x,t,tau,om]
tau=1;om=0.25;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]□0,x[0]□0,x'[0]□1},
x[t],t]
Plot[x[t]/.sol,{t,0,50},PlotRange→All]
```

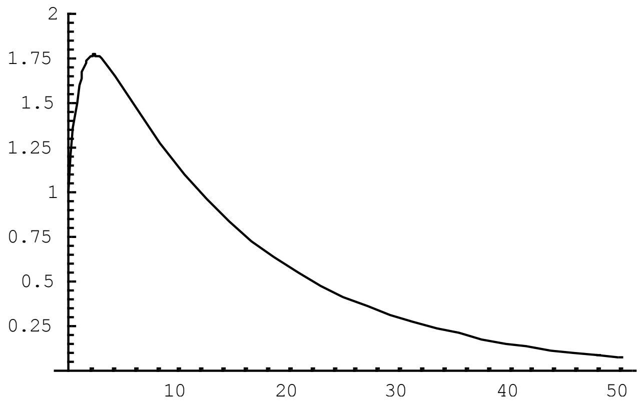
```
{{x[t] → □ □^(1. t) (□1.1547 □^(0.0669873 t) □1.1547 □^(0.933013 t))}}
```



For x(0)=1 m and $\dot{x}(0)=1m/s$ one finds:

```
ClearAll[x,t,tau,om]
tau=1;om=0.25;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]□0,x[0]□1,x'[0]□1},
x[t],t]
Plot[x[t]/.sol,{t,0,50},PlotRange→{0,2}]
```

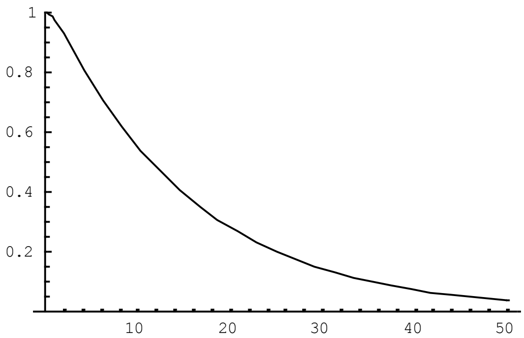
```
{{x[t] → □ □^(1. t) (□1.23205 □^(0.0669873 t) □2.23205 □^(0.933013 t))}}
```



For x(0)=1 m and $\dot{x}(0)=0m/s$ one finds:

```
ClearAll[x,t,tau,om]
tau=1;om=0.25;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]□0,x[0]□1,x'[0]□0},
x[t],t]
Plot[x[t]/.sol,{t,0,50},PlotRange→{0,1}]
```

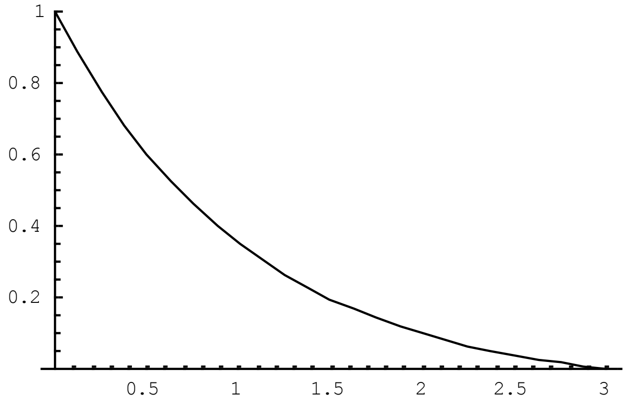
```
{{x[t] → □ □^(1. t) (□0.0773503 □^(0.0669873 t) □1.07735 □^(0.933013 t))}}
```



For $x(0)=1$ m and $\dot{x}(0) = -1 m/s$ one finds:

```
ClearAll[x,t,tau,om]
tau=1;om=0.25;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]==0,x[0]==1,x'[0]==-1},x[t],t]
Plot[x[t]/.sol,{t,0,3},PlotRange→{0,1}]
```

$\{\{x[t] \to e^{-1.t}(1.07735\, e^{0.0669873\, t} - 0.0773503\, e^{0.933013\, t})\}\}$



Conclusion: The solution essentially depends on the initial conditions (of course, also on the equation and the coefficients).

b) $\frac{1}{\tau} = \omega_0$ **(critical damping). Equal roots**

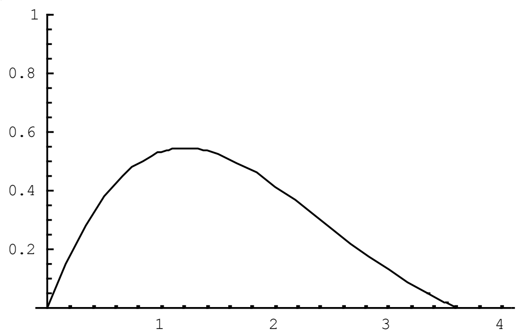
Analytic solution:

$$x(t) = (At + B)e^{-t/\tau} \qquad \text{(O14)}$$

For $x(0)=0$ and $\dot{x}(0) = 1 m/s$ one finds:

```
ClearAll[x,t,tau,om]
tau=1;om=1;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]==0,x[0]==0,x'[0]==1},x[t],t]
Plot[x[t]/.sol,{t,0,4},PlotRange→{0,1}]
```

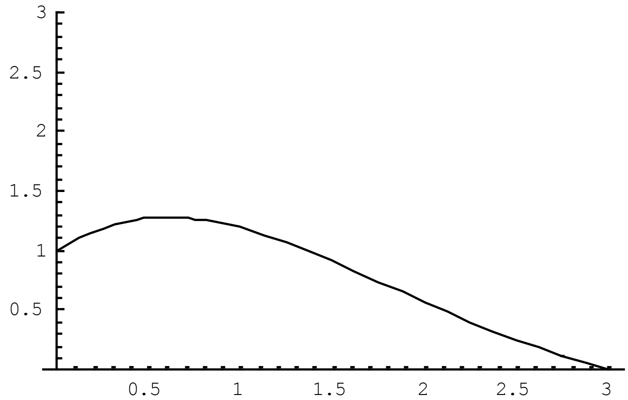
$\left\{\left\{x[t] \to \frac{2\, e^{-t/2}\, \mathrm{Sin}\left[\frac{\sqrt{3}\, t}{2}\right]}{\sqrt{3}}\right\}\right\}$



For $x(0)$=1 m and $\dot{x}(0) = 1 m/s$ one finds:

```
ClearAll[x,t,tau,om]
tau=1;om=1;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]==0,x[0]==1,x'[0]==1},
x[t],t]
Plot[x[t]/.sol,{t,0,3},PlotRange→{0,3}]
```

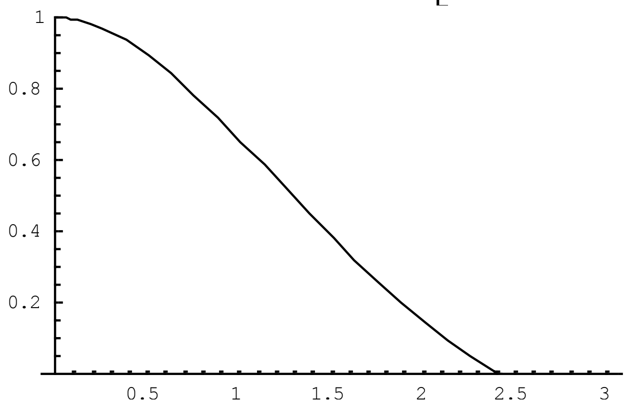
$$\left\{\left\{x[t] \to e^{-t/2}\left(\mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \sqrt{3}\,\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]\right)\right\}\right\}$$



For $x(0)$=1 m and $\dot{x}(0) = 0 m/s$ one finds:

```
ClearAll[x,t,tau,om]
tau=1;om=1;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]==0,x[0]==1,x'[0]==0},
x[t],t]
Plot[x[t]/.sol,{t,0,3},PlotRange→{0,1}]
```

$$\left\{\left\{x[t] \to \frac{1}{3} e^{-t/2}\left(3\,\mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \sqrt{3}\,\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]\right)\right\}\right\}$$

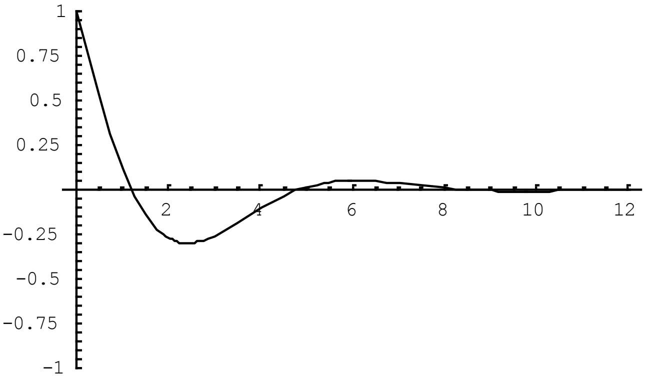


For $x(0)$=1 m and $\dot{x}(0) = -1 m/s$ one finds:

```
ClearAll[x,t,tau,om]
tau=1;om=1;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]==0,x[0]==1,x'[0]==-
1},x[t],t]
```

```
Plot[x[t]/.sol,{t,0,12},PlotRange→{-1,1}]
```

$$\left\{\left\{x[t] \to \frac{1}{3} e^{-t/2}\left(3\,\text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \sqrt{3}\,\text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]\right)\right\}\right\}$$



c) $\frac{1}{\tau} < \omega_0$ **small damping, complex conjugated roots**

Analytic solution:

$$x(t) = e^{-t/\tau}\left(A\sin(\omega t) + A\cos(\omega t)\right) = Ce^{-t/\tau}\sin(\omega t + \varphi_0) \qquad \text{(O15)}$$

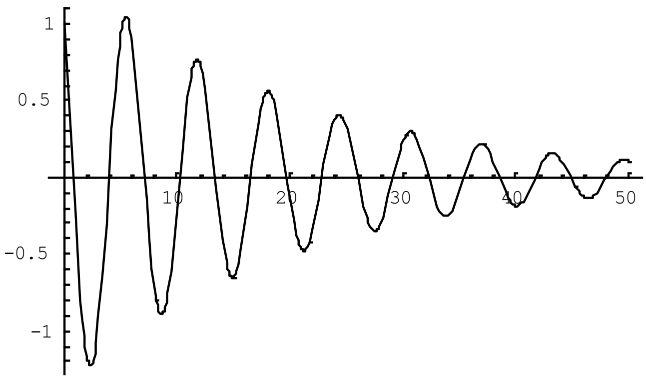
with the effective angular frequency

$$\omega = \sqrt{\omega_0^2 - \left(\frac{1}{\tau}\right)^2} \qquad \text{(O16)}$$

For *x*(0)=0 and $\dot{x}(0) = 1m/s$ **(**and for other conditions as well) **one finds:**

```
ClearAll[x,t,tau,om]
tau=10;om=1;
sol=DSolve[{x''[t]+x'[t]/tau+om^2*x[t]==0,x[0]==1,x'[0]==-1},x[t],t]
Plot[x[t]/.sol,{t,0,50},PlotRange→All]
```

$$\left\{\left\{x[t] \to \frac{1}{21} e^{-t/20}\left(21\,\text{Cos}\left[\frac{\sqrt{399}\,t}{20}\right] - \sqrt{399}\,\text{Sin}\left[\frac{\sqrt{399}\,t}{20}\right]\right)\right\}\right\}$$



This is the most important situation: the point oscillates, but with diminishing amplitude. The movement is not perfectly periodic, but only *quasi-periodic*, with a *quasi-period* defined as:

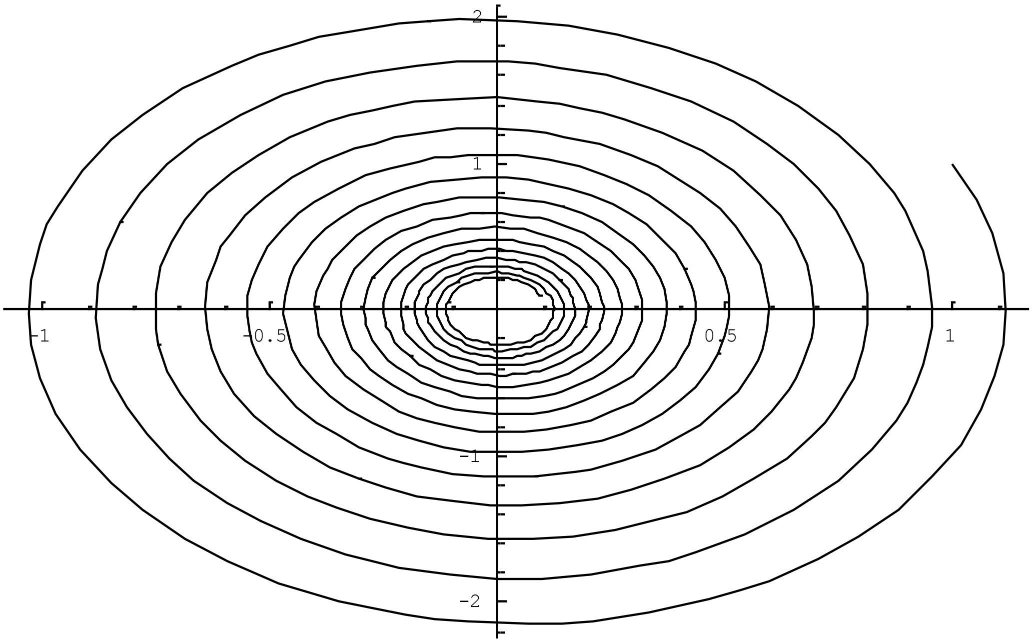
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \left(\frac{1}{\tau}\right)^2}} \tag{O17}$$

The logarithmic decrement is defined by:

$$\delta = \ln\frac{x(t)}{x(t+T)} = \frac{T}{\tau} \tag{O18}$$

An attenuated oscillator in phase space goes inward:

```
ParametricPlot[{xsol[t],xsol'[t]},{t,0,15*Pi}]
```



Question: What would represent a spiral going outward ?