Mathematical appendix 1-2012

1. Taylor series

We are interested in approximations arround a point x_0 of a function $f(x)$ by a polynomial of degree $n \geq 0$, eventually by a series.

n=0 what is the approximation ? obviously $f(x_0)$

n=1 what is the approximation ? A curve should be locally substituted by a straight line – the tangent –

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$ (figure)

n=2 what is the approximation ? A curve should be locally substituted by a parabola – (figure)

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2$$

n=3 what is the approximation ? A curve should be locally substituted by a 3-rd degree polynomial (figure) –

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3$$

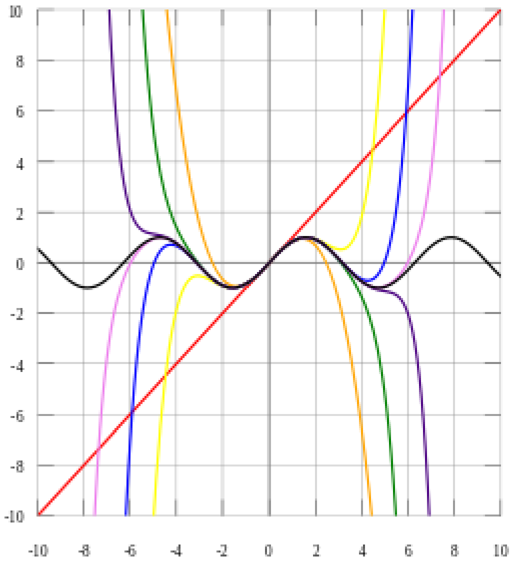
$n=\infty$

$$\boxed{f(x) \approx \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n}$$ Taylor series

The expansion is true also for complex variable x, if derivatives exist.

Examples. (http://en.wikipedia.org/wiki/Taylor_series)

Sin$x$ in x=0



As the degree of the Taylor polynomial rises, it approaches the correct function. This image shows $\sin x$ and Taylor approximations, polynomials of degree 1, 3, 5, 7, 9, 11 and 13.

Exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

Natural logarithm:

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } -1 \le x < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for } -1 < x \le 1$$

Finite geometric series:

$$\frac{1-x^{m+1}}{1-x} = \sum_{n=0}^{m} x^n \quad \text{for } x \ne 1 \text{ and } m \in \mathbb{N}_0$$

Infinite geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Variants of the infinite geometric series:

$$\frac{x}{x-1} = \sum_{n=0}^{\infty} x^{-n} \quad \text{for } |x| > 1$$

$$\frac{x^m}{1-x} = \sum_{n=m}^{\infty} x^n \quad \text{for } |x| < 1 \text{ and } m \in \mathbb{N}_0$$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n \quad \text{for } |x| < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{for } |x| < 1$$

Square root:

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2(4^n)} x^n = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \dots \quad \text{for } |x| \le 1$$

Trigonometric functions

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

See also e.g. http://www.efunda.com/math/taylor_series/taylor_series.cfm

2. Important application: exponential with imaginary exponent

z is real

$$e^{iz} = 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \dots + \frac{(iz)^n}{n!} + \dots =$$

$$1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots \quad + \qquad (*)$$

$$i\left(\frac{z}{1!}-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right) \qquad (**)$$

The line (*) derived gives minus the bracket from line (**) **Hence (*) is cos function**

The bracket from line (**) derived gives line (*) **Hence (**) is sin function**

As may be seen from the expansions page 3 above. An important relation (Euler):

$$e^{iz} = \cos z + i \sin z$$

Applications: ***Euler relations***

$$e^{ia}\cdot e^{ib} = e^{i(a+b)} = \cos(a+b)+i\sin(a+b)=(\cos a + i\sin a)(\cos b + i\sin b)$$
$$= \cos a\cos b - \sin a\sin b + i(\sin a\cos b + \sin b\cos a)$$

$$e^{ia}\cdot e^{-ib} = e^{i(a-b)} = \cos(a-b)+i\sin(a-b)=(\cos a + i\sin a)(\cos b - i\sin b)$$
$$= \cos a\cos b + \sin a\sin b + i(\sin a\cos b - \sin b\cos a)$$

$$\cos a = \frac{e^{ia}+e^{-ia}}{2} \qquad \sin a = \frac{e^{ia}-e^{-ia}}{2i}$$

3. Partial derivatives

Simple examples.

4. Vector algebra

Simple examples

5.