Assumptions needed to employ differential and integral calculus

1. One can't use finite laws of the form

$$x=Vt$$
$$I=Q/t$$

unless some quantities are constant.
Such laws are correct for constant velocity or for constant intensity. If they are not, relations are true only on the average. This is not a proper description of reality.
Think of examples.

2. **1D example.** The alternative is to reduce intervals until variations are small and could be neglected. We define in this way instantaneous velocity or instantaneous intensity. We assume small enough time intervals Δt such as a constant velocity v must be defined. The small distance the body moves in this small time is $\Delta x = v\Delta t$. The velocity is constant if $\Delta t \to 0$. Such an "infinitesimal" variation is denoted by dt and the corresponding distance is dx. Therefore

$$dx = vdt \tag{1}$$

The total distance the body travels during the time interval $t_2 - t_1$ is the integral

$$x_{12} = \int_{t_1}^{t_2} v(t)dt \tag{2}$$

Eq. (1) gives the definition of instantaneous velocity:

$$v = \lim \frac{\Delta x}{\Delta t} \text{ for } \Delta t \to 0, \qquad \text{or} \qquad v = \frac{dx}{dt} \equiv x'(t) \tag{3}$$

Relations (1-3) assume that functions are gentle enough to perform derivatives or integrals.

Counter-example:The Weierstrass function, defined by $\sum_{k=1}^{\infty} \mathrm{Sin}[\pi k^2 x]/(\pi k^2)$, *which is continuousbutdifferentiableonly on a set of points ofmeasure zero. The figure below represents an approximation of the real function, the infinity being replaced by 100.*

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Plot[Sum[Sin[Pi*k^2*x]/(Pi*k^2),{k,1,100} ],{x,0,1}]
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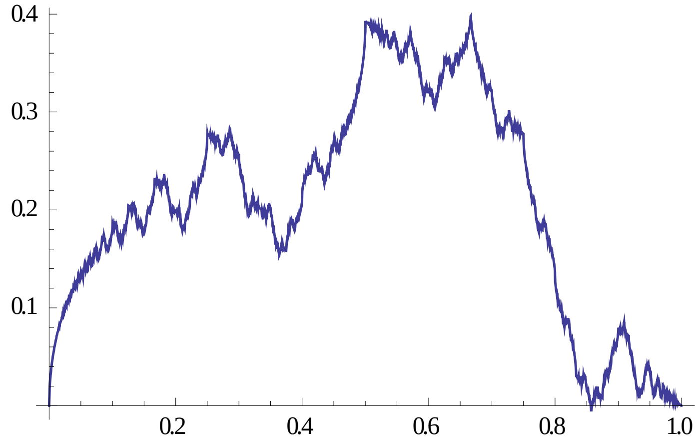


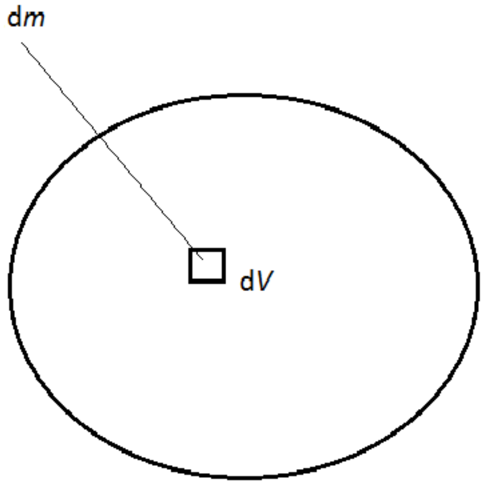
3. **3D example.** The density of a homogeneous body is given by

$$\rho = \frac{m}{V} \qquad (4)$$

The mass results as $m = \rho V$ (4')

If the body is non-homogeneous, we divide it in tiny volumes and define a *local density* $\rho(x,y,z) = \frac{dm}{dV} = \frac{dm}{dxdydz}$ **(!)**

(4'')



The mass of this small volume d*V* is

$$dm = \rho dV = \rho dxdydz \qquad (5)$$

And the total mass is

$$m = \iiint_{volume} \rho dV = \iiint_{volume} \rho dxdydz \qquad \textbf{(!!)} \qquad (6)$$