

2. Quantum physics

2.1. Quantum behaviour

Consequences of the above chapter:

- i) Microsystems behave as waves and as particles as well – **wave-particle dualism**
- ii) States of microsystems may be expressed as linear combinations of other states – combination or **superposition principle**
- iii) Energies, angular momenta and perhaps other physical quantities may be quantized – **quantification**
- iv) Measurements modify the state of a system in a non-controlled manner; conjugated quantities can not be defined with arbitrary precision – **uncertainty principle**. These conjugated quantities are e.g. *the coordinate and the associated linear momentum*.

2.2. Quantum postulates

Remark: the following postulates do not form a complete family of logically consistent propositions from which all the theory could be deduced. They are rather a way to construct quantum world and are based on experiments as well as on inferences tested a posteriori.

1. States of a system are elements of a linear space called the state space. They are symbolized: by Greek letters φ , ψ , Ψ , Φ ; by Greek letters with variable specifications, $\Psi(x, y, z, t)$; by symbols in brackets, $\langle \Phi(x, y, z) |$, $|\psi(\vec{r}, t)\rangle$, $|3, 2, 0\rangle$. In the last forms they are known as *bras* and *kets*.

The interpretation of the wave function: its modulus squared gives the probability density for the system to be in a certain region, or to have a certain momentum:

$$|\psi(\vec{r})|^2 = \langle \psi^* | \psi \rangle = \text{density of probability to have the system around the position vector } \vec{r}, \text{ i.e. between } x \text{ and } x+dx, y \text{ and } y+dy, z \text{ and } z+dz. \quad (\text{QM2.1})$$

The state functions are usually normalized:

$$\int |\psi(\vec{r})|^2 dx dy dz = 1 \quad (\text{QM2.2})$$

State functions must be *finite, continuous, with continuous derivatives* (with one exception, points in which the potential has infinite discontinuities).

2. To each physical quantity \mathcal{A} is associated a certain Hermitian operator \hat{A} . The measurement of the physical quantity \mathcal{A} in the state ψ is represented by the

action of the operator \hat{A} on this state. Such operators have real eigenvalues and their eigenvectors form a base of the space. They are called *observables*.

As the result of the measurement the state of the system changes in a chaotic way. A new measurement completed immediately after the first leaves the state unchanged. The results of the measurements are the eigenvalues of the observables.

3. The limit of quantum relations should be the classical results (*correspondence principle*)

4. *Fundamental commutators:*

$$[\hat{x}, \hat{p}_x] = i\hbar, [\hat{y}, \hat{p}_y] = i\hbar, [\hat{z}, \hat{p}_z] = i\hbar \quad (\text{QM2.3})$$

All the other are zero.

2.3. The Schrödinger equation

If the system evolves freely, without any perturbation – such as measurements – the state at a later moment $t + dt$ is perfectly defined by the state at a previous moment t .

Eventually we get the Schrödinger equation:

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = \hat{H}|\Psi(t)\rangle \quad (\text{QM2.5})$$

One may show that $\hat{H} = \hat{E}_k + \hat{U}$ is the *Hamiltonian* of the system. Assume $|\psi_n(\vec{r}, t)\rangle$ is an eigenfunction of \hat{H} , belonging to the eigenvalue E_n : $\hat{H}|\psi_n(\vec{r}, t)\rangle = E_n|\psi_n(\vec{r}, t)\rangle$. For such a state Eq. (QM2.5) gives:

$$i\hbar \frac{d|\psi_n(\vec{r}, t)\rangle}{dt} = \hat{H}|\psi_n(\vec{r}, t)\rangle = E_n|\psi_n(\vec{r}, t)\rangle$$

with a simple solution

$$|\psi_n(\vec{r}, t)\rangle = \exp\left[-\frac{i}{\hbar} E_n t\right] |\psi_n(\vec{r})\rangle \quad (\text{QM2.6})$$

The temporal dependence factorizes out. Such states are known as *stationary states*, because their modulus squared is constant. They are the eigenfunctions of the equation with eigenvalues for the Hamilton operator. The *time-independent*

Schrödinger equation $\hat{H}\psi = E\psi$ is written as: $\left(\frac{\hat{p}^2}{2m} + \hat{U}\right)\psi = E\psi$, or, for potential

energy depending only on coordinates:

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + U(\vec{r}) \right) \psi = E \psi \quad (\text{QM2.7})$$

2.4. One-dimensional examples

2.4.1. The rectangular potential barrier

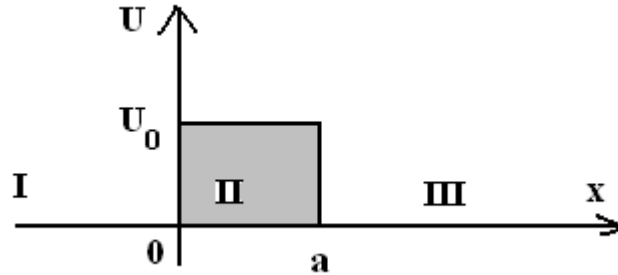


Fig. 1. 1D square potential barrier

The Schrödinger equation (QM2.7) writes

$$\psi'' + \frac{2m}{\hbar^2} (E - U(x)) \psi = 0$$

In the three regions we write:

$$\psi_I'' + \frac{2m}{\hbar^2} E \psi_I = 0 \quad \text{with solution} \quad \psi_I(x) = A_1 e^{ikx} + B_1 e^{-ikx} \quad (\text{QM2.8'})$$

$$\psi_{II}'' + \frac{2m}{\hbar^2} (E - U_0) \psi_{II} = 0 \quad \text{with solution} \quad \psi_{II}(x) = A_2 e^{k_2 x} + B_2 e^{-k_2 x}$$

$$(\text{QM2.8''}) \quad \psi_{III}'' + \frac{2m}{\hbar^2} E \psi_{III} = 0 \quad \text{with solution} \quad \psi_{III}(x) = A_3 e^{ikx}$$

(QM2.8''')

(see lectures for details). Here

$$k_1^2 = \frac{2mE}{\hbar^2} \quad k_2^2 = \frac{2m}{\hbar^2} (U_0 - E) \quad (\text{QM2.9})$$

Continuity conditions:

$$\psi_I(0) = \psi_{II}(0), \quad \psi_{II}(a) = \psi_{III}(a), \quad \psi_I'(0) = \psi_{II}'(0), \quad \psi_{II}'(a) = \psi_{III}'(a)$$

The interesting quantity is the *transparency*

$$D = \frac{A_3^* A_3}{A_1^* A_1}$$

which measures the probability to find the particle on the right if initially it moves towards the barrier from $x = -\infty$. If we could neglect the term proportional to $e^{-k_2 a}$ with regard to the term proportional to $e^{k_2 a}$ the final result is:

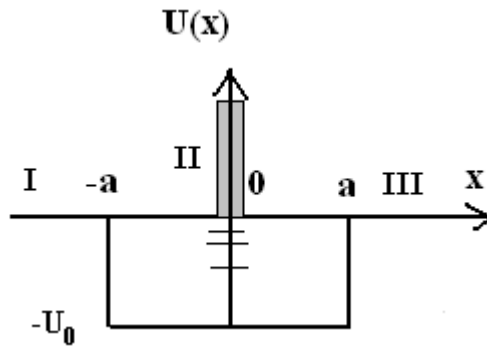
$$D = 16 \frac{E(U_0 - E)}{U_0^2} \exp \left[-\frac{2}{\hbar} \sqrt{2m(U_0 - E)} a \right] \quad (\text{QM2.10})$$

For the general barrier described by a function $U(x)$, the result is (see lectures):

$$D \approx 16 \frac{E(U_0 - E)}{U_0^2} \exp \left[-\frac{2}{\hbar} \int_{x_1}^{x_2} \sqrt{2m(U_0 - E)} dx \right] \quad (\text{QM2.10}')$$

The factor $D_0 = 16 \frac{E(U_0 - E)}{U_0^2}$ is close to the unity.

Potential well



For $E > 0$ the energetic spectrum is continuous. We are interested in the situation $E < 0$. Inside the well $-a < x < a$ the Schrödinger equation is identical with that in (QM2.8'), outside it is the same as that from (QM2.8''). As the potential is symmetric, solutions may be symmetric (even) or anti-symmetric (odd).

Even solutions:

$$\psi_1^{(+)} = C e^{+k_1 x}$$

$$\psi_2^{(+)} = A \cos k_2 x$$

$$\psi_3^{(+)} = C e^{-k_1 x}$$

Odd solutions

$$\psi_1^{(-)} = C e^{+k_1 x}$$

$$\psi_2^{(-)} = B \sin k_2 x$$

$$\psi_3^{(-)} = -C e^{-k_1 x}$$

with notations from (QM2.9): $k_1^2 = \frac{2m|E|}{\hbar^2}$, $k_2^2 = \frac{2m}{\hbar^2}(U_0 - |E|)$. Continuity

conditions in $-a/2$ and $a/2$ give eventually:

$$\text{for even solutions - } \tan k_2 a = \frac{k_1}{k_2}$$

for odd solutions – $\cot k_2 a = -\frac{k_1}{k_2}$

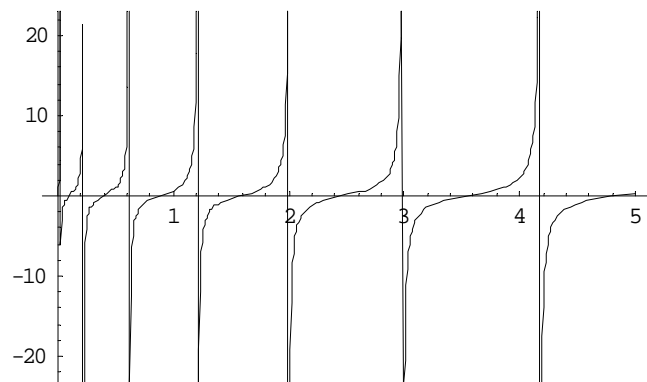
Both relations may be written in the form $\tan k_2 a = \frac{2k_1 k_2}{k_1^2 - k_2^2}$. But k_1 and k_2 are

both functions of the energy E . The last equation gives the possible values for this energy, which is *quantified*. The program below shows the graphs:

```

ClearAll[x,k1,k2,e,u0]
a=10;u0=5;
k1[e_]=Sqrt[e]
k2[e_]=Sqrt[u0-e]
f[e_]=Tan[k1[e]*a]
g[e_]=2*k1[e]*k2[e]/((k1[e])^2-(k2[e])^2)
Plot[f[e],{e,0,u0}]

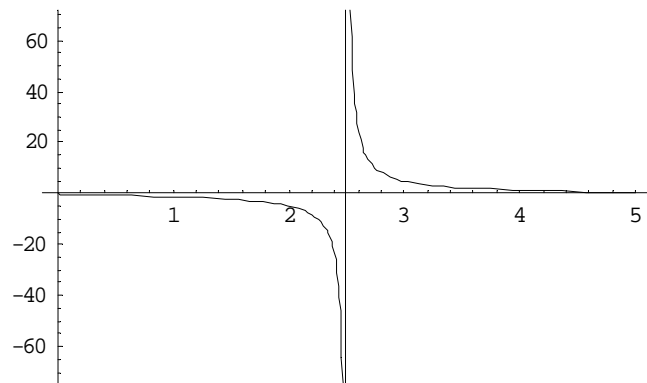
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Plot[g[e],{e,0,u0}]

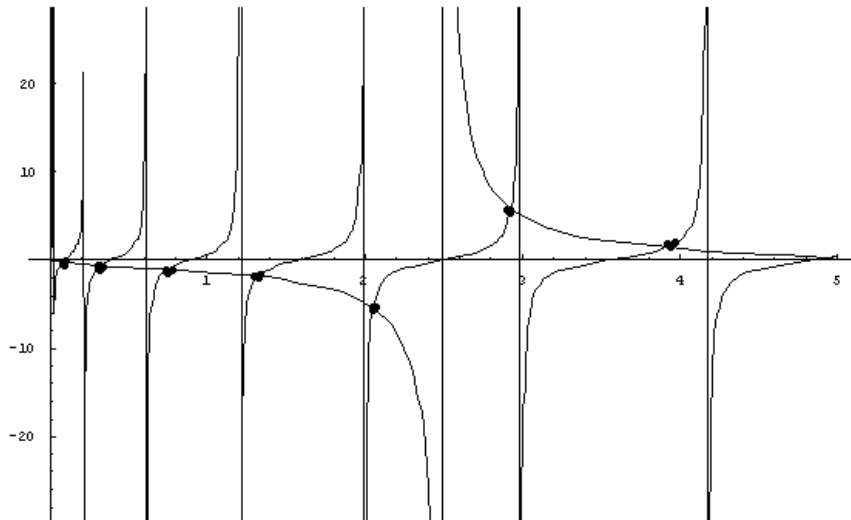
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Points give the allowed energies. In the example there are only 7 discrete energy levels in the well. They are called *bound states*.

2.4.2. Quantum harmonic oscillator

$$\hat{H}\psi(x) = -\frac{\hbar^2}{2m}\psi''(x) + \frac{m\omega^2 x^2}{2}\psi(x) = E\psi(x)$$

With notations $\varepsilon = \frac{\hbar\omega}{2}$ and $x = \sqrt{\frac{\hbar}{m\omega}}\xi = \frac{\xi}{x_0}$:

$$\frac{d^2\psi}{dx^2} + (\varepsilon - \xi^2)\psi = 0$$

Finite solutions depend on a quantum number $n = 0, 1, 2, 3, \dots$

$$\psi_n(\xi) = C_n H_n(\xi) e^{-\xi^2/2} \quad (\text{QM2.11})$$

with

$$H_n(\xi) = (2\xi)^n - \frac{n(n-1)}{1!}(2\xi)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2\xi)^{n-4} - \dots \quad (\text{QM2.12})$$

the Hermite polynomials.

Energies are quantified as

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \quad (\text{QM2.13})$$

To compute the normalization constants, use the *normalization condition*:

$$\int_{-\infty}^{\infty} |\psi(\xi)|^2 d\xi = 1 \quad (\text{QM2.14})$$

One finds $C_n = [2^n n! \sqrt{\pi}]^{-1/2}$, so the first three normalized eigenfunctions are:

$$\psi_0(x) \equiv |0\rangle = \frac{1}{(\sqrt{\pi}x_0)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

$$\psi_1(x) \equiv |1\rangle = \frac{1}{(2\sqrt{\pi}x_0)^{1/2}} \frac{2x}{x_0} \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

$$\psi_2(x) \equiv |2\rangle = \frac{1}{(8\sqrt{\pi}x_0)^{1/2}} \left[4\left(\frac{x}{x_0}\right)^2 - 2\right] \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

2.5. Electron in a central field

2.5.1. The Schrödinger equation in spherical coordinates

$$-\frac{\hbar^2}{2m_0} \Delta \psi(\vec{r}) + U(r)\psi(\vec{r}) = E\psi(\vec{r}) \quad (\text{QM2.15})$$

Go to spherical variables and get for the Laplace operator:

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] = \quad (\text{QM2.16})$$

$$= \Delta_r + \frac{1}{r^2} \Delta_{\theta, \varphi}$$

$$(15) \text{ becomes: } \left(\Delta_r + \frac{1}{r^2} \Delta_{\theta, \varphi} \right) \psi(r, \theta, \varphi) + k^2(r) \psi(r, \theta, \varphi) = 0 \quad (\text{QM2.15'})$$

with

$$k^2(r) = \frac{2m_0}{\hbar^2} [E - U(r)]$$

Separation of variables: $\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$.

The simplest equation is the one for $\Phi(\varphi)$:

$$\frac{d^2 \Phi}{d\varphi^2} + m_l^2 \Phi = 0$$

with solutions $\Phi(\varphi) = Ce^{im\varphi}$. The periodicity condition $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ gives

$$m_l = 0, \pm 1, \pm 2, \pm 3 \dots \quad (\text{QM2.17})$$

The number m_l is called *the magnetic quantum number*.

The equation in $\Theta(\theta)$ is difficult:

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left(\lambda - \frac{m^2}{\sin^2 \theta} \right) \Theta = 0, \quad (\text{QM2.18})$$

with m the *magnetic quantum number* and λ another quantum number, coming from equation:

$$\Delta_{\theta, \varphi} Y(\theta, \varphi) + \lambda Y(\theta, \varphi) = 0 \quad (\text{QM2.19})$$

Conditions for continuous, single-valued and finite solutions imply

$$\lambda = l(l+1), \quad l=0, 1, 2, \dots \quad (\text{QM2.20})$$

Another condition is $|m_l| \leq l$ (QM2.21)

Solutions of Eq. (QM2.19) with the above limitations are known as *spherical harmonics*. Some expressions of the spherical functions are:

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}} \\ Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos \theta & Y_{1\pm 1} &= \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\varphi} \\ Y_{20} &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) & Y_{2\pm 1} &= \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\varphi} & Y_{2\pm 2} &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\varphi} \end{aligned}$$

It turns out that spherical functions Y_{lm} are eigenfunctions of the operator square of the angular momentum $\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2$. The exact eigenvalue equation is:

$$\hat{l}^2 Y_{lm}(\theta, \varphi) = l(l+1)\hbar^2 Y_{lm}(\theta, \varphi), \quad (\text{QM2.22})$$

$$\text{with } l=0, 1, 2, \dots \quad \text{and } m_l = 0, \pm 1, \pm 2, \dots, \pm l \quad (\text{QM2.23})$$

The magnitude of the angular momentum takes only quantized values equal to

$$\sqrt{l(l+1)}\hbar \quad l=0, 1, 2, \dots \quad (\text{QM2.24})$$

For the operator \hat{l}_z one finds the eigenvalue equation

$$\hat{l}_z \psi = l_z \psi$$

with eigenvalues $l_z = m_l \hbar$, $m_l = 0, \pm 1, \pm 2, \dots, \pm l$ (QM2.25)

2.5.2. The Hydrogen atom (and hydrogen-like atoms)

For more graphs see e.g. <http://undergrad-ed.chemistry.ohio-state.edu/H-AOs/>

The potential energy is $U(r) = \frac{Ze^2}{4\pi\epsilon_0 r}$. The radial equation could be solved

analytically and the finite, continuous solutions are the Laguerre polynomials. They depend on a single *principal quantum number* $n=1, 2, 3, \dots$. If we put together all the preceding results, we see that the eigenfunctions $\psi_{nlm_l}(r, \theta, \varphi) \equiv |n, l, m_l\rangle$ depend on three different quantum numbers:

The principal q.n. $n=1, 2, 3, \dots$

The azimuthal q.n. $l=0, 1, 2, \dots, n-1$

The magnetic q.n. $m_l = 0, \pm 1, \pm 2, \dots, \pm l$

See attached files H eigenfunc 1 & 2.

2.5.3. Characterization of stationary states of H atoms.

Each state of a H atom is described by three quantum numbers, n , l , and m_l . Therefore the notation $\psi_{nlm}(r, \theta, \varphi) \equiv |n, l, m\rangle$. These q.n. fix the quantified values of the energy (see below), the magnitude of the angular momentum and of its projection on the Oz axis, Eqs. (24, 25). Energy is given by the expression from Bohr's theory:

$$E_n = -\frac{1}{n^2} \frac{m_0 e^4}{2\hbar^2} = -\frac{1}{n^2} \frac{m_0 e^4}{2\hbar^2 (4\pi\epsilon_0)^2} = -\frac{13.6}{n^2} \text{ eV} = -\frac{Ry}{n^2} \quad (\text{QM2.26})$$

Here
$$Ry = \frac{m_0 e^4}{2\hbar^2 (4\pi\epsilon_0)^2} = \frac{1}{2} m_0 c^2 \left(\frac{e^2}{4\pi\epsilon_0 \hbar c} \right)^2 = \alpha^2 \frac{1}{2} m_0 c^2 \quad (\text{QM2.26'})$$

is the *rydberg*. The factor in the bracket is a dimensionless quantity called the *fine structure constant*:

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137} \quad (\text{QM2.27})$$

Because the energy does not depend on l and m_l each level is degenerated and to each value of the principal q.n. n correspond n^2 different states with $l=0, 1, 2, \dots, n-1$ and $m_l = 0, \pm 1, \pm 2, \dots, \pm l$.

Spectral notation: $l=0$ s-states
 $l=1$ p-states

$l=2$ d -states

$l=3$ f -states

Possible states: $1s$; $2s, 2p$; $3s, 3p, 3d$; $4s, 4p, 4d, 4f$;

The three observables \hat{H}, \hat{l}^2 and \hat{l}_z form a complete set of commutative operators and their eigenvalues give the maximum information we could have on the state.

See the applet <http://www.falstad.com/qmatom/> for atomic orbitals and 1-D periodic potentials.

2.6. Emission and absorption of radiation

2.6.1 *Quantum transitions.* Eigenstates of the Hamiltonian described above are *stationary*, i.e. a system being in such a state will remain there. But quantum systems experience transitions from one stationary state to another and doing so they emit or absorb energy, usually under the form of photons. That means the microsystem has not the simple Hamiltonian we used before; it is *perturbed* by some extra terms describing the interaction with other systems. As a result transitions occur.

There is a supplementary condition: the perturbation must contain a frequency given by Bohr's condition:

$$\omega_{mn} = \frac{E_m^0 - E_n^0}{\hbar} \quad (\text{QM2.28'})$$

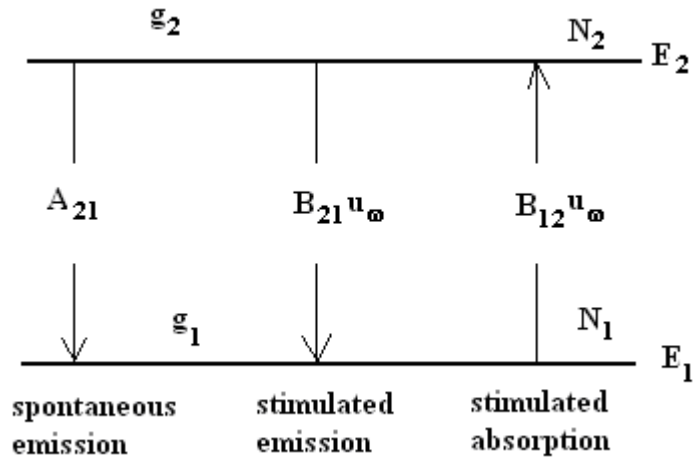
For each perturbation we have to compute a certain quantity which is a function of the initial and the final state, as well as of the perturbation. If this quantity is zero, the transition is *forbidden*. If it is non-zero, the transition is *allowed*. These computations lead to *selection rules* controlling the variations of the quantum numbers.

2.6.2. *Selection rules.* The most encountered perturbations are given by electric or magnetic interactions. They could be classified following the type of field: dipole, quadrupole,... The most intense elmg transitions in atoms and molecules are the *dipolar-electric transitions*. The selection rules for such transitions are:

$$\Delta l = \pm 1 \quad \Delta m_l = 0, \pm 1 \quad (\text{QM2.29})$$

There are no selection rules for the principal q.n. Selection rules explain thoroughly atomic and molecular spectra, as well as nuclear gamma spectra.

2.6.3. *Einstein's theory of emission and absorption.* Assume identical atoms in an external elmg field. To make simple, atoms have but two energy levels with energies E_1 and E_2 . Levels have degeneracy g_1 and g_2 . The number of atoms are N_1 and N_2 . The field may have many frequencies but we suppose that the Bohr frequency of the transition (QM2.28') is present. The spectral density of energy of the elmg field is $u_\omega(T) \equiv u(\omega_{21})$



Three processes: *spontaneous* and *stimulated emission* and *stimulated absorption*. Denote by A_{21} the probability for an atom to suffer a spontaneous emission in a time Δt . For stimulated processes such a probability should be proportional to $u(\omega_{21})$ and also to the corresponding *Einstein coefficient* B_{21} or B_{12} . Assume for ease that $g_1 = g_2$. The *detailed balance* equation is:

$$[A_{21} + B_{21}u(\omega_{21})]N_2\Delta t = B_{12}u(\omega_{21})N_1\Delta t \quad (*)$$

Hence

$$\frac{N_1}{N_2} = \exp\left[\frac{\hbar\omega_{21}}{k_B T}\right] = \frac{A_{21}}{B_{21}u(\omega_{mn})} + \frac{B_{21}}{B_{12}}$$

This relation holds even for $T \rightarrow \infty$, hence $B_{21} = B_{12}$.

At equilibrium use the Boltzmann distribution:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left[-\frac{E_2 - E_1}{k_B T}\right] = \frac{g_2}{g_1} \exp\left[-\frac{\omega_{21}}{k_B T}\right] = \exp\left[-\frac{\omega_{21}}{k_B T}\right]$$

The final result is the Planck relation:

$$u(\omega_{21}) = \frac{A_{21}}{B_{21}} \frac{1}{\exp\left[\frac{\hbar\omega_{21}}{k_B T}\right] - 1}$$

It follows also that

$$\frac{A_{21}}{B_{21}} = \frac{\omega_{21}^3}{\pi^2 c^3} \quad (\text{QM2.30})$$

The spontaneous coefficient A has s^{-1} as MU. If during dt a number dN_2 atoms suffer transitions from the state 2 to the state 1, one can write: $-dN_2 = A_{21}N_2 dt$, hence

$$N_2(t) = N_2(0)e^{-t/\tau} \quad (\text{QM2.31})$$

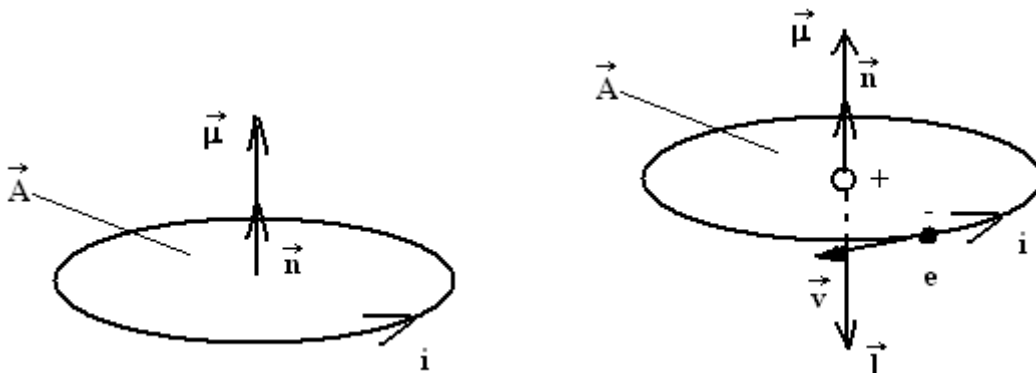
The duration τ is the average time the atom stays in the excited state, *the lifetime* of the state.

2.7. Magnetic interaction. Electronic spin

2.7.1. *The orbital magnetic moment of the electron.* If an atom is put in an external magnetic field its spectrum becomes more complex. Many spectral lines split into several closed components with different polarizations. This effect was discovered by Zeeman in 1897. Some lines split in just three components; this is the *normal effect* and is explained classically. Indeed in classical electromagnetism each magnetic moment interacts with a magnetic field through an energy term

$$E_{mgn} = -\vec{\mu} \cdot \vec{B} \quad (\text{QM2.32})$$

Here $\vec{\mu}$ is the magnetic moment defined in analogy with the same quantity for a small loop as $\vec{\mu} = i\vec{A}$ (see figure below)



The vector $\vec{A} = A\vec{n}$ is associated with the sense of the current i by the usual rule. If the current is given by an electron revolving around the nucleus at a distance r with speed v , its magnetic moment would be: $\mu = -\frac{e}{2\pi r/v} \pi r^2 = -\frac{evr}{2} = -\frac{em_0vr}{2m_0} = -\frac{el}{2m_0}$, with l the angular momentum. One may write also:

$$\vec{\mu} = -\frac{e}{2m_0}\vec{l} = -\gamma_{orb}\vec{l} \quad (\text{QM2.33})$$

The quantity γ_{orb} is the *orbital gyromagnetic factor*. The magnetic energy is

$$E_{mgn} = -\vec{\mu} \cdot \vec{B} = \frac{e}{2m_0}\vec{l} \cdot \vec{B} = \gamma_{orb}\vec{l} \cdot \vec{B} \quad (\text{QM2.34})$$

In quantum mechanics:

$$\hat{H}_{mgn} = -\hat{\mu} \cdot \vec{B} = \frac{e}{2m_0}\hat{l} \cdot \vec{B} = \gamma_{orb}\hat{l} \cdot \vec{B} \quad (\text{QM2.34}')$$

Assume the magnetic field along Oz and use results from 2.5 to find:

$$\hat{H}_{mgn} = \gamma_{orb}\hat{l}_z \cdot B = m \frac{e\hbar}{2m_0} B = \{0, \pm 1, \pm 2, \dots\} \frac{e\hbar}{2m_0} B \quad (\text{QM2.35})$$

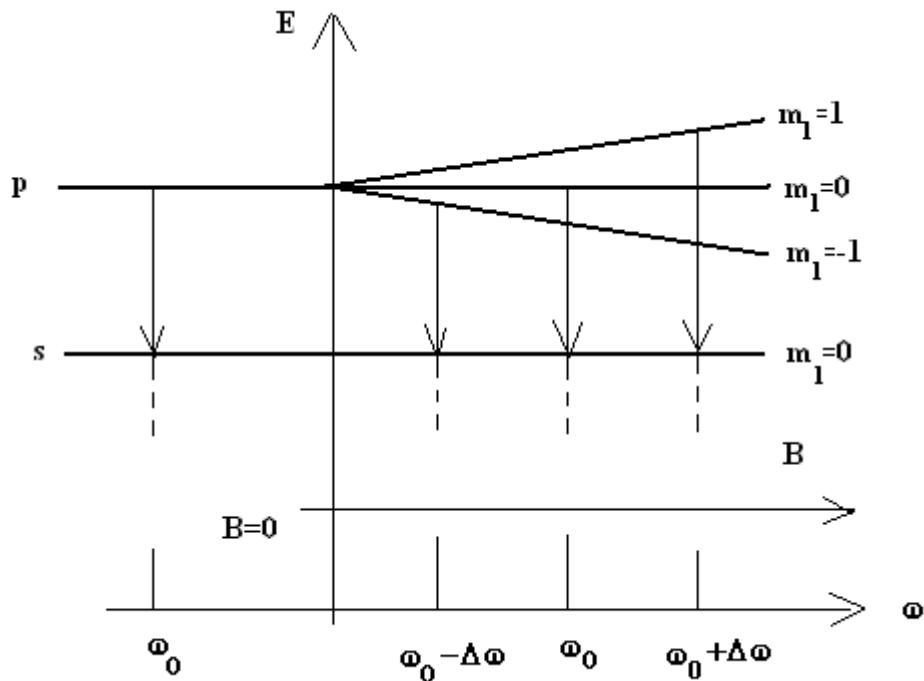
The quantity (*taken from CODATA*, a site giving last values for physical constants)

$$\mu_B = \frac{e\hbar}{2m_0} \cong 9.27400915(23) \times 10^{-24} \text{ J}\cdot\text{T}^{-1} \quad (\text{QM2.36})$$

is the *Bohr magneton*. For a H atom the quantified energy of a level becomes

$$E_n \rightarrow E_n + E_{mgn} = -\frac{Ry}{n^2} + m_l \mu_B B \quad (\text{QM2.37})$$

The splitting of atomic levels and of spectral lines is shown below for a $p \rightarrow s$ transition.



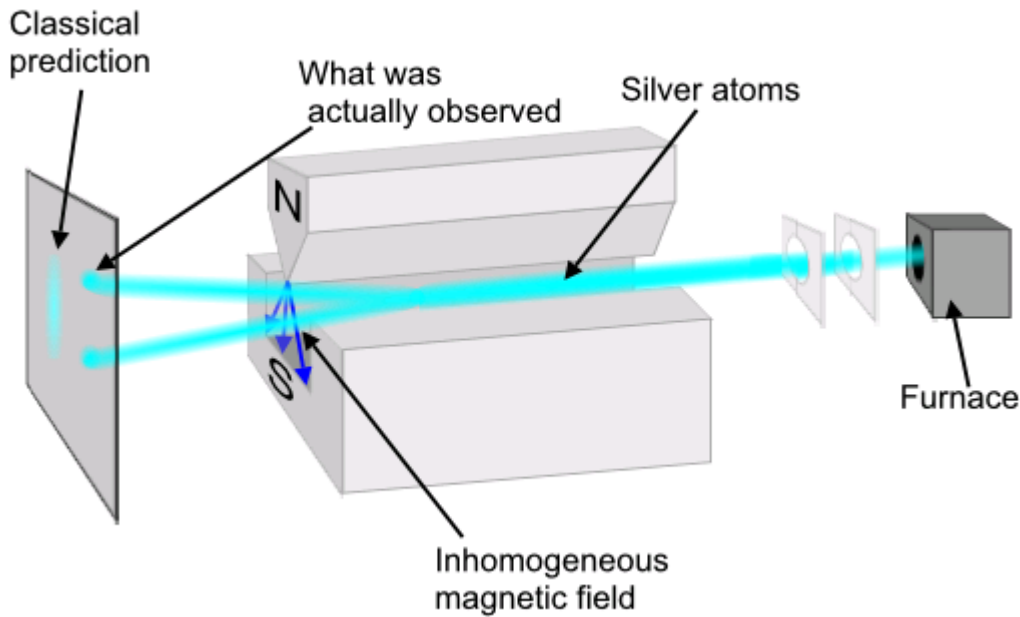
The deviation $\hbar\Delta\omega = \mu_B B = \frac{e\hbar}{2m_0} B$ is the value found experimentally.

Exercise: show that even for more complex spectra (e.g. for a $d \rightarrow p$ transition) one get a three-fold splitting of spectral lines.

2.7.2. *Electronic spin.* So in magnetic field each spectral line should split in three components. But this is not true: one finds doublets, quartets ... This is the *anomalous Zeeman effect*. The explanation is given by the existence of *electronic spin*. This hypothesis set forth by Uhlenbeck and Goudsmit in 1925 is based on spectroscopy and on the experiment of Stern and Gerlach made in 1922. The sketch of the experiment is shown below (from Wikipedia).

Exercise: read and learn the explanation in

<http://www.upscale.utoronto.ca/GeneralInterest/Harrison/SternGerlach/SternGerlach.html>, or <http://hyperphysics.phy-astr.gsu.edu/hbase/spin.html#c2>



The inhomogeneous magnetic field exercise upon a beam of atoms a vertical force along the z direction given by $F = \vec{\mu}_{atom} \cdot \frac{\partial \vec{B}}{\partial z} = \mu_{atom} \frac{\partial B}{\partial z} \cos(\vec{\mu}_{atom}, \vec{B})$. Atoms (Ag) were chosen to have no magnetic moment, so the force ought to be zero. However deviations were observed. Assumptions were made to take into account some nonzero orbital magnetic moment. Classical theory allows any angle between the relevant vectors hence the classical prediction in the drawing. An orbital magnetic moment should conduct to an *odd* number of lines. What was actually observed was just a pair of spots as if the magnetic moment may have just two orientations, up and down. The corresponding value for the intrinsic – or spin – quantum number have to be $s = \frac{1}{2}$, so as to get $2s + 1 = 2$ possible orientation. The values oh the corresponding magnetic moment are $\pm \mu_B$. The *spin hypothesis* says:

a) The electron has an intrinsic angular momentum, called *spin*, described by operators $\hat{s}^2, \hat{s}_x, \hat{s}_y, \hat{s}_z$. They fulfill the usual commutation relations of angular moments and commute with all coordinate and momentum operators.

b) The operator \hat{s}^2 has but one eigenvalue

$$s = \frac{1}{2} : \hat{s}^2 \psi = s(s + 1) \hbar^2 \psi = \frac{3}{4} \hbar^2 \psi \quad (\text{QM2.38'})$$

Its projection along Oz axis has two eigenvalues:

$$\hat{s}_z \psi = m_s \hbar \psi \quad \text{with} \quad m_s = \pm \frac{1}{2}$$

(QM2.38'')

c) To this spin angular momentum is associated a magnetic moment with quantified projections $\mu_{sz} = \pm \mu_B$.

The gyromagnetic ratio for the spin is twice as big as for the orbital case (gyromagnetic spin anomaly):

$$\gamma_{orb} = \frac{e}{2m_0} \quad \gamma_s = \frac{\left(\frac{e\hbar}{2m_0}\right)}{\hbar/2} = \frac{e}{m_0} = g_s \frac{e}{2m_0} = g_s \gamma_{orb} \quad (\text{QM.2.39})$$

Consequences: doublets, quadruplets ... are allowed and explained; complex spectra are explained; the periodic table is explained.

Examples of Zeeman splitting (from <http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/zeeman.html>)

"Normal" Zeeman effect

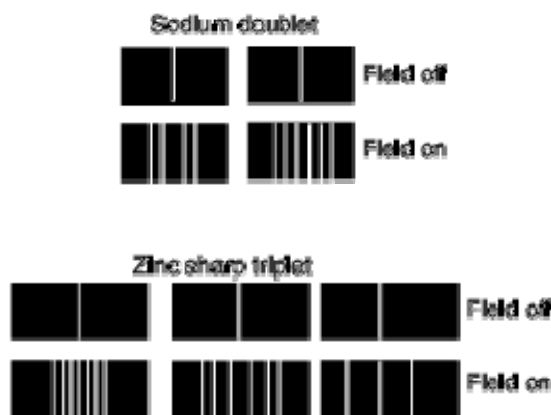
This type of splitting is observed with hydrogen and the zinc singlet.



This type of splitting is observed for spin 0 states since the spin does not contribute to the angular momentum.

"Anomalous" Zeeman effect

When electron spin is included, there is a greater variety of splitting patterns.



3. Quantum statistics

3.1. Types of quantum statistics

It turns out that every microscopic particle (elementary particles, atoms, molecules) has spin. For a complex particle the spin builds up from the individual spins of the components. Pauli shown that there are two very different families of particles:

- Particles with integer spins, $s=0, 1, 2, 3, \dots$. They are called *bosons* and obey the Bose-Einstein statistics: at equilibrium the average number of particles with energy ε_i is given by:

$$\langle n_i \rangle_{B-E} = \frac{g_i}{\exp\left[\frac{\varepsilon_i - \mu_{B-E}}{k_B T}\right] - 1} \quad (\text{QM2.40})$$

Here g_i is the degeneracy and μ_{B-E} the chemical potential. Examples: photons, an even number of electrons or of protons. Bosons may occupy in any number a given state. Their functions of state are symmetric to the particle permutations:

$$\psi(1, 2, \dots, j, \dots, k, \dots) = +\psi(1, 2, \dots, k, \dots, j, \dots)$$

- Particles with half-integer spins, $s=1/2, 3/2, 5/2, \dots$. They are called *fermions* and obey the Fermi-Dirac statistics: at equilibrium the average number of particles with energy ε_i is given by:

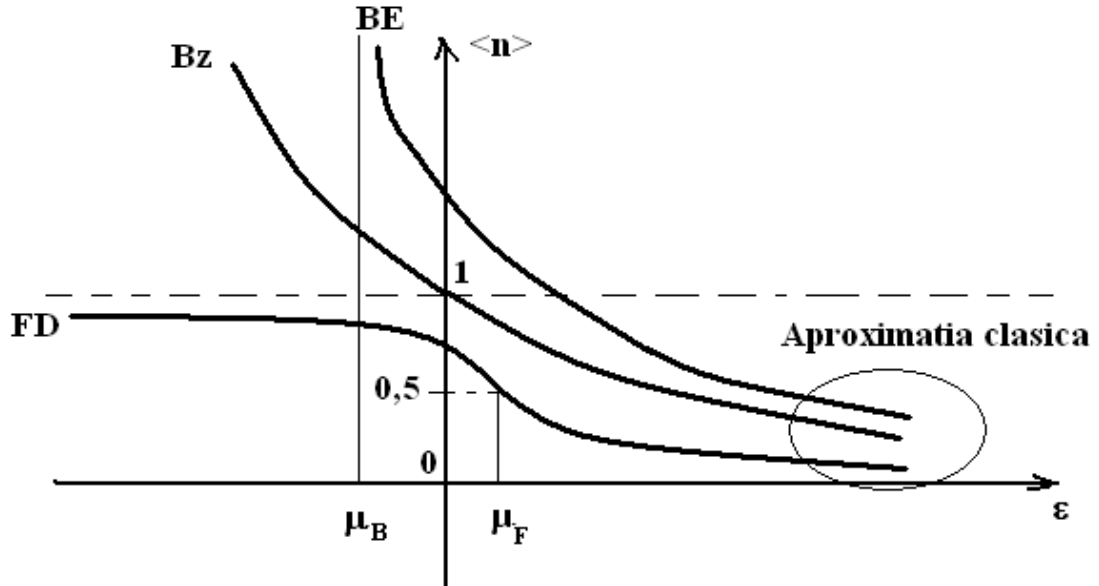
$$\langle n_i \rangle_{F-D} = \frac{g_i}{\exp\left[\frac{\varepsilon_i - \mu_F}{k_B T}\right] + 1}$$

$$(\text{QM2.41})$$

Here g_i is the degeneracy and μ_F the chemical potential or the *Fermi level*. Examples: electrons, protons, neutrons, an odd number of fermions. Fermions could be at most one in a state (the *Pauli exclusion principle*). Their functions of state are anti-symmetric to the particle permutations:

$$\psi(1, 2, \dots, j, \dots, k, \dots) = -\psi(1, 2, \dots, k, \dots, j, \dots)$$

A comparison between classical (Boltzmann) and quantum statistics is given in the following figure. At large values of the quantity at the exponent $\frac{\epsilon_i - \mu_F}{k_B T} \gg 1$ both quantum statistics have the same classical limit.



3.2. Lasers

According to (QM.2.30) the spontaneous emission is too large in the optical field and a generator of stimulated radiation is difficult to build in this domain. For such a generator to exist the stimulated power emitted must be larger than the absorbed power supplemented by the dissipated power: $P_e > P_a + P_{ext}$. The power dissipates mainly by photons going out of the device. The rate of emitted and absorbed photons are (see (*) p. 11)

$$N_2 B u(\omega) = N_2 W_{21} \quad \text{and} \quad N_1 B u(\omega) = N_1 W_{12} \quad (\text{QM2.42})$$

$$W_{21} = W_{12} = B_{21} u(\omega) \quad (\text{QM2.43})$$

are the *stimulated rates* per atom for stimulated processes.

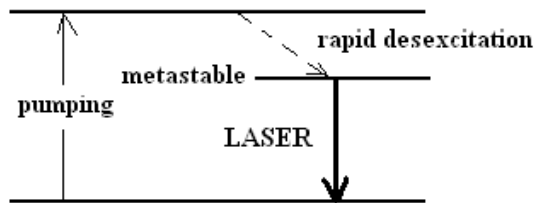
$$\frac{dI}{dz} = \sigma I (N_2 - N_1) \quad (\text{QM2.47})$$

As a rule the intensity reduces along the path because $N_2 < N_1$; the radiation is absorbed by the material. *To amplify the radiation we need population inversion*

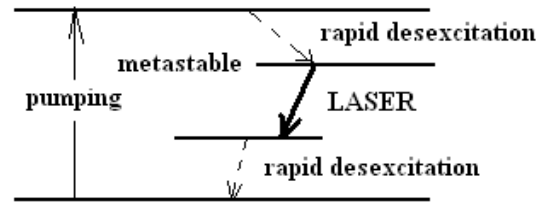
$N_2 > N_1$. The Boltzmann relation (no degeneracy) is $\frac{N_2}{N_1} = \exp\left[-\frac{E_2 - E_1}{k_B T}\right]$,

therefore the amplification is obtained for *negative absolute temperatures* $T < 0$ K.

Two level-schemes for laser are presented below.

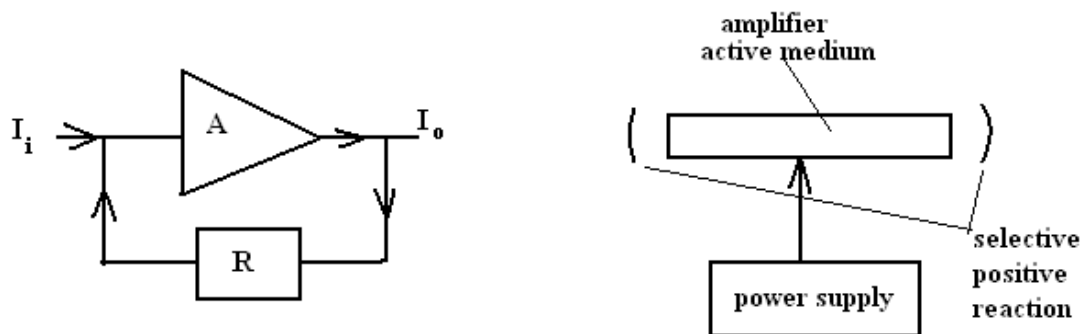


3-level scheme



4-level scheme

Laser block-scheme.



Without reaction $I_o = AI_i$. The reaction sends back a part of the output intensity

$$I_o = A(I_i + RI_o) \quad \rightarrow \quad I_o = \frac{A}{1 - RA} I_i \quad (\text{QM2.48})$$

In general the amplification A and the reaction R depend on the frequency. If for a certain frequency $RA=1$, the oscillation condition is satisfied and the system becomes a generator of stimulated light.

3.3. Electrons in solids. Semiconductors.

3.3.1. Band structure of solids

From Wikipedia:

“The electrons of a single isolated atom occupy [atomic orbitals](#), which form a discrete set of [energy](#) levels. If several atoms are brought together into a molecule, their atomic orbitals split, as in a [coupled oscillation](#). This produces a number of [molecular orbitals](#) proportional to the number of atoms. When a large number of atoms (of order $\times 10^{20}$ or more) are brought together to form a solid, the number of orbitals becomes exceedingly large, and the difference in energy between them becomes very small, so the levels may be considered to form continuous *bands* of energy rather than the discrete energy levels of the atoms in isolation. However, some intervals of energy contain no orbitals, no matter how many atoms are aggregated, forming *band gaps*.

Within an energy band, energy levels are so numerous as to be a near continuum. First, the separation between energy levels in a solid is comparable with the energy that electrons constantly exchange with [phonons](#) ([atomic vibrations](#)). Second, it is comparable with the energy uncertainty due to the [Heisenberg uncertainty principle](#), for reasonably long intervals of time. As a result, the separation between energy levels is of no consequence.”

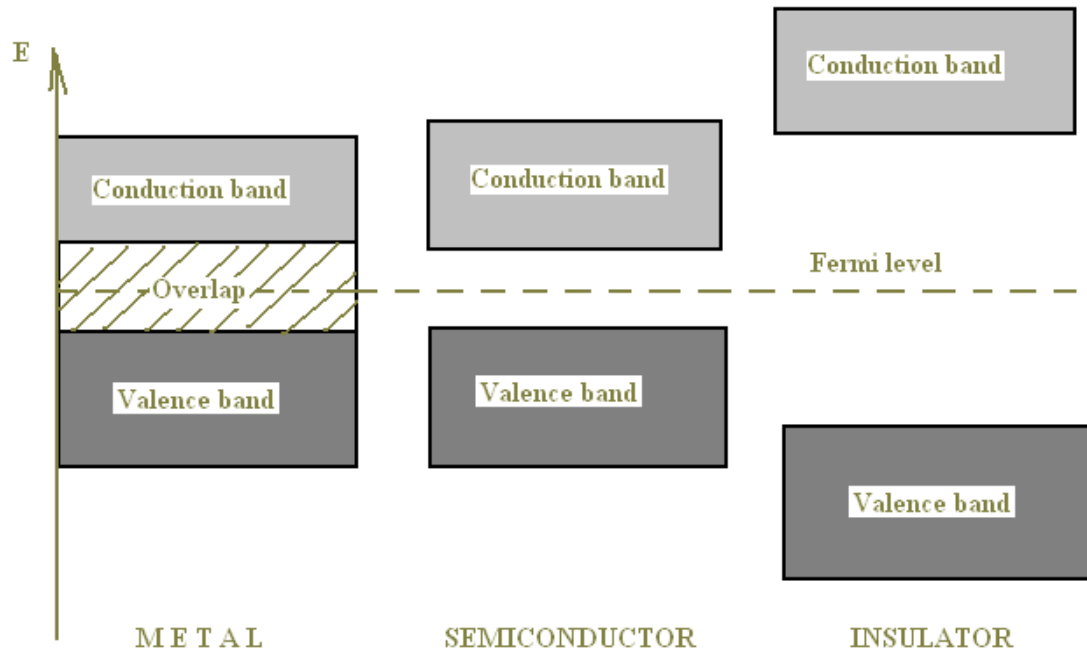
Any solid has a large number of bands. In theory, it can be said to have infinitely many bands (just as an atom has infinitely many energy levels). However, all but a few lie at energies so high that any electron that reaches those energies escapes from the solid. These bands are usually disregarded.

Bands have different widths, based upon the properties of the atomic orbitals from which they arise. Also, allowed bands may overlap, producing (for practical purposes) a single large band.

Figure 1 shows a simplified picture of the bands in a solid that allows the three major types of materials to be identified: metals, semiconductors and insulators.

[Metals](#) contain a band that is partly empty and partly filled regardless of temperature. Therefore they have very high conductivity.

The lowermost, almost fully occupied band in an [insulator](#) or [semiconductor](#), is called the [valence band](#) by analogy with the [valence electrons](#) of individual atoms. The uppermost, almost unoccupied band is called the [conduction band](#) because only when electrons are excited to the conduction band can current flow in these materials. The difference between insulators and semiconductors is only that the forbidden [band gap](#) between the valence band and conduction band is larger in an insulator, so that fewer electrons are found there and the [electrical conductivity](#) is lower. Because one of the main mechanisms for electrons to be excited to the conduction band is due to thermal energy, the conductivity of semiconductors is strongly dependent on the temperature of the material.



This band gap is one of the most useful aspects of the band structure, as it strongly influences the electrical and optical properties of the material. Electrons can transfer from one band to the other by means of [carrier generation and recombination](#) processes. The band gap and defect states created in the band gap by [doping](#) can be used to create [semiconductor devices](#) such as [solar cells](#), [diodes](#), [transistors](#), [laser diodes](#), and others.” (End of Wikipedia)

General statement: states above the Fermi level are mostly empty; states below are typically filled with electrons. (see lectures)

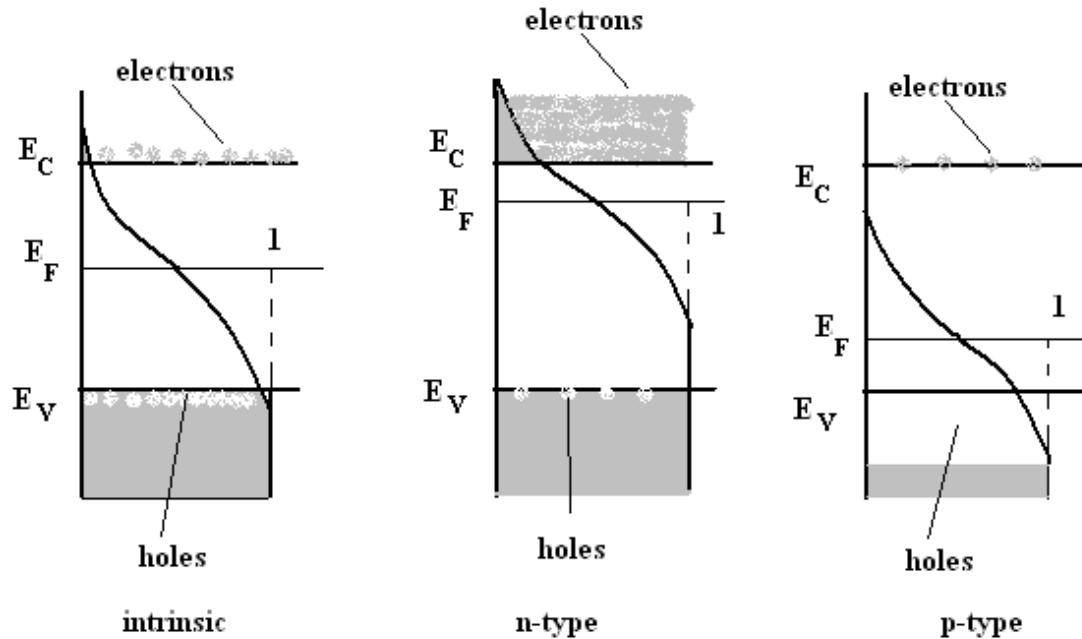
Metals: The Fermi level is in the middle of allowed bands. Electrons easily acquire energy from an external electric field and may participate to the conduction. The density of carriers is $10^{22} - 10^{23} \text{ cm}^{-3}$. It does not vary much with T .

Semiconductors. The Fermi level is in the middle of the forbidden gap, whose energy is $E_g \approx 1 \text{ eV}$. Carriers are electrons in the conduction band (CB) and holes in the valence band (VB). To participate to conduction an e^- must “jump” from VB to CB and for this it needs energy, e.g. thermal energy. At room temperature $k_B T = \frac{1}{40} \text{ eV}$.

The carrier density varies mainly exponentially with temperature, between 10^{12} and 10^{20} cm^{-3} .

Insulators. The Fermi level is in the middle of the forbidden gap, whose energy is $E_g \approx 5 \text{ eV}$. At usual T an insulator does not conduct electrical currents.

3.3.2. Carrier statistics in semiconductors



For intrinsic scnd the Fermi level is roughly in the middle of the gap. For doped scnd the Fermi level moves towards the CB (n-type) or VB (p-type) as in Fig. above. In this figure we see that the concentrations of carriers in both bands depend very much on the position of the Fermi level:

- they are equal in the intrinsic case
- the e^- concentration in CB is much bigger that holes concentration in the VB for *n*-type semiconductors
- the concentration pf holes in the VB is much bigger that e^- concentration in the CB for *p*-type semiconductors

Electrons in CB are “quasi-particles” which may be different from free electrons. In particular their mass depends to a great extent on the particular structure of the CB. We may find (sometimes even in the same material) light electrons, with *effective mass* m^* smaller than the mass of free electrons m_0 and heavy electrons, with $m^* > m_0$. The same is true for the holes in the VB.

In *nondegenerate semiconductors*: the Fermi level is well into the forbidden band and the Fermi distribution is approximated with the Boltzmann's:

$$E_C - E_F > 5k_B T, \quad E_F - E_V > 5k_B T \quad (\text{QM2.49})$$

It can be shown that in this case the densities of electrons in the CB and holes in the VB are given by:

$$n = 2 \left(\frac{2\pi m_n^* k_B T}{h^2} \right)^{3/2} \exp \left[\frac{E_F - E_C}{k_B T} \right] = N_C(T) \exp \left[\frac{E_F - E_C}{k_B T} \right] \quad (\text{QM2.50}')$$

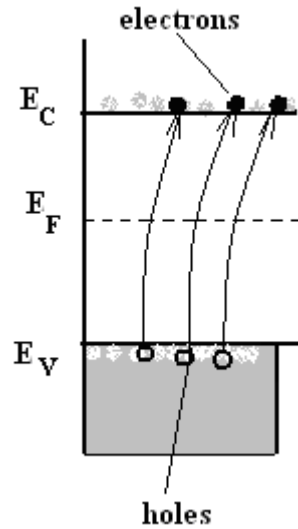
$$p = 2 \left(\frac{2\pi m_p^* k_B T}{h^2} \right)^{3/2} \exp \left[\frac{E_V - E_F}{k_B T} \right] = N_V(T) \exp \left[\frac{E_V - E_F}{k_B T} \right] \quad (\text{QM2.50}'')$$

In such nondegenerate semiconductors the product np is constant:

$$np = N_C N_V \exp \left[-\frac{E_C - E_V}{k_B T} \right] = N_C N_V \exp \left[-\frac{E_g}{k_B T} \right] = n_i^2 \quad (\text{QM2.51})$$

where E_g is the energy gap. Quantities N_C and N_V are the *effective concentrations* of e^- in the CB and of holes in the VB. In Si at 300 K they are of the order of 10^{19} cm^{-3} .

a) *Intrinsic semiconductors.*

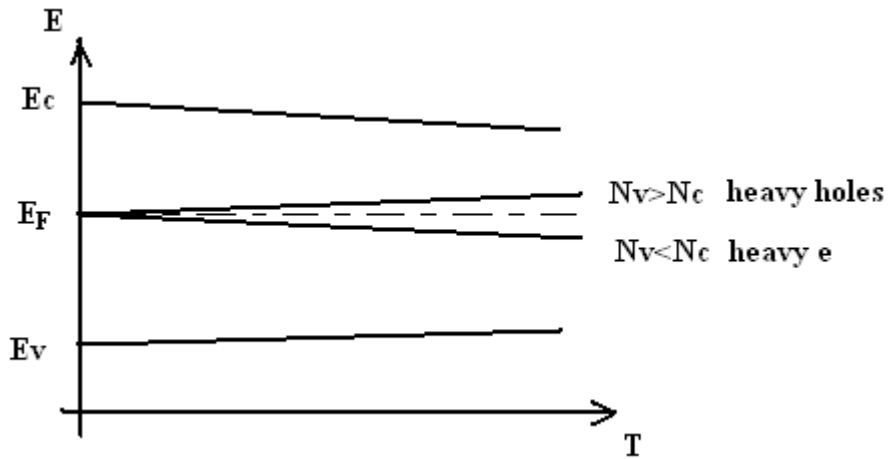


The *neutrality condition*: $n = p$ writes with (QM2.50):

$$N_C \exp \left[\frac{E_F - E_C}{k_B T} \right] = N_V \exp \left[\frac{E_V - E_F}{k_B T} \right]. \text{ The position of the Fermi level is:}$$

$$E_F = \frac{E_V + E_C}{2} + \frac{k_B T}{2} \ln \frac{N_V}{N_C} = \frac{E_V + E_C}{2} + \frac{3k_B T}{4} \ln \frac{m_p^*}{m_n^*} \quad (\text{QM2.52})$$

The position of E_F when T varies is shown below:



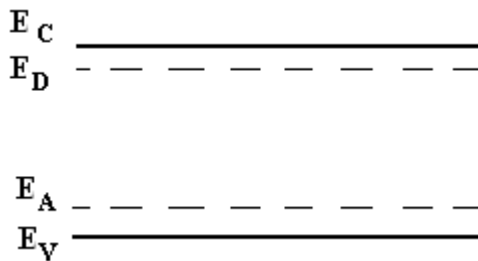
The intrinsic concentration is

$$n_i = \sqrt{N_C N_V} \exp\left[-\frac{E_g}{2kT}\right] \quad (\text{QM2.53})$$

Exercise: Draw the graph of $\ln(n_i) = f(1/T)$. What quantity could be computed from the slope of this graph ?

b) Impurified semiconductors

The most general case: donors and acceptors *compensated semiconductor*.



Taking into account spin degeneration of e on the impurities levels and the fact that a ionized donor level means the absence of an e the *neutrality condition* is roughly:

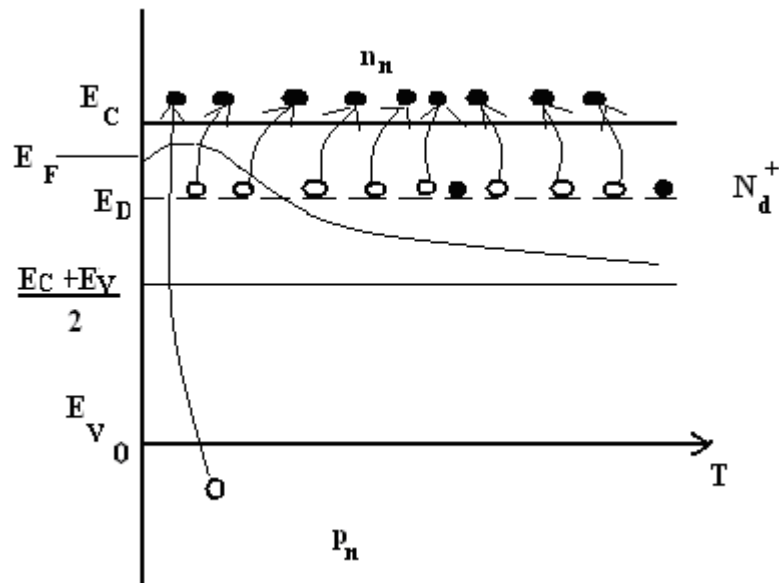
No. of e in CB + No. ionized acceptors = No. of holes in VB + No. ionized donors

$$n + N_A^- = p + N_D^+ \quad (\text{QM2.54})$$

(The numbers are actually concentrations)

b1) Type-n semiconductors.

Impurities from the Vth group e.g. P, As, have extra levels in the gap close to E_C as in the drawing below:



The *neutrality condition*: $n_n = N_d^+ + p_n$. N_d^+ is the concentration of ionized donors.

These ionized donors have lost their electrons, so their concentration is (up to a degeneracy factor 2 in the denominator)

$$N_d^+ = \frac{N_d}{2 \exp\left[\frac{E_F - E_D}{k_B T}\right] + 1}$$

Neglecting the minority concentration p_n one finds:

$$N_C \exp\left[\frac{E_F - E_C}{kT}\right] = \frac{N_D}{\exp\left[\frac{E_F - E_D}{kT}\right] + 1}.$$

Denoting $x = \exp\left[\frac{E_F}{kT}\right]$

$$2N_C \exp\left[-\frac{E_D + E_C}{kT}\right] x^2 + N_C \exp\left[-\frac{E_C}{kT}\right] x - N_D = 0, \text{ or}$$

$$x^2 + \frac{\exp\left[\frac{E_D}{kT}\right]}{2} x - \frac{N_D}{2N_C} \exp\left[\frac{E_C + E_D}{kT}\right] = 0$$

Keeping only the positive root:

$$x = \frac{\exp\left[\frac{E_D}{kT}\right]}{4} \left(-1 + \sqrt{1 + \frac{8N_D}{N_C} \exp\left[\frac{E_C - E_D}{kT}\right]} \right) \quad \text{or}$$

$$E_F = E_D + kT \ln \left\{ \frac{1}{4} \left(-1 + \sqrt{1 + \frac{8N_D}{N_C} \exp\left[\frac{E_C - E_D}{kT}\right]} \right) \right\} \quad (\text{QM2.55})$$

Limit Cases.

A) *Very low T* (a few K). Condition: $\frac{8N_D}{N_C} \exp\left[\frac{E_C - E_D}{kT}\right] \gg 1$. We neglect the unity in (QM2.55) and find

$$E_F = \frac{E_C + E_D}{2} + \frac{kT}{2} \ln \left[\frac{N_D}{2N_C} \right] \quad (\text{QM2.55}')$$

At $T=0$ K the Fermi level is at half the distance between E_C and E_D . When T grows E_F approaches the conduction band and later drops as in the figure above.

The e^- concentration is

$$n_n(\text{very low } T) \approx \sqrt{\frac{N_C N_D}{2}} \exp\left[-\frac{E_C - E_D}{2kT}\right] \quad (\text{QM2.56}')$$

The graph $\ln n = f\left(\frac{1}{T}\right)$ is virtually a straight line with the slope $-\frac{E_C - E_D}{2kT}$.

This is a method to find the position of the donor level below E_C .

Question: what is the situation in an intrinsic semiconductor?

B) *Low temperatures* (from several tens of K to room temperatures). Condition:

$$\frac{8N_D}{N_C} \exp\left[\frac{E_C - E_D}{kT}\right] \ll 1, \text{ hence } N_C \gg 8N_D. \text{ We develop the square root}$$

and eventually find

$$E_F = E_C + kT \ln \left[\frac{N_D}{N_C} \right] \quad (\text{QM2.55}'')$$

The argument of ln is negative, so the Fermi level comes close to the middle of the gap. The e^- concentration is

$$n_n(\text{low } T) \approx N_C \exp\left[\ln\left(\frac{N_D}{N_C}\right)\right] = N_D \quad (\text{QM2.56''})$$

This is the *depletion region* when all the donors are ionized and the intrinsic generation is low. The majority carriers have an almost constant concentration; the concentration of the minority carriers grows rapidly with T :

$$p_n = \frac{n_i^2}{n_n} = \frac{n_i^2}{N_D} = N_C N_V \exp\left[-\frac{E_g}{2kT}\right]$$

C) *High temperatures.* We go back to the neutrality condition and assume all the donors are ionized $n_n = N_D + p_n = N_D + \frac{n_i^2}{n_n}$. The positive solution is:

$$n_n = \frac{N_D}{2} \left(1 + \sqrt{1 + \frac{4n_i^2}{N_D^2}} \right), \text{ and hence } p_n = \frac{2n_i^2}{N_D \left(1 + \sqrt{1 + \frac{4n_i^2}{N_D^2}} \right)}$$

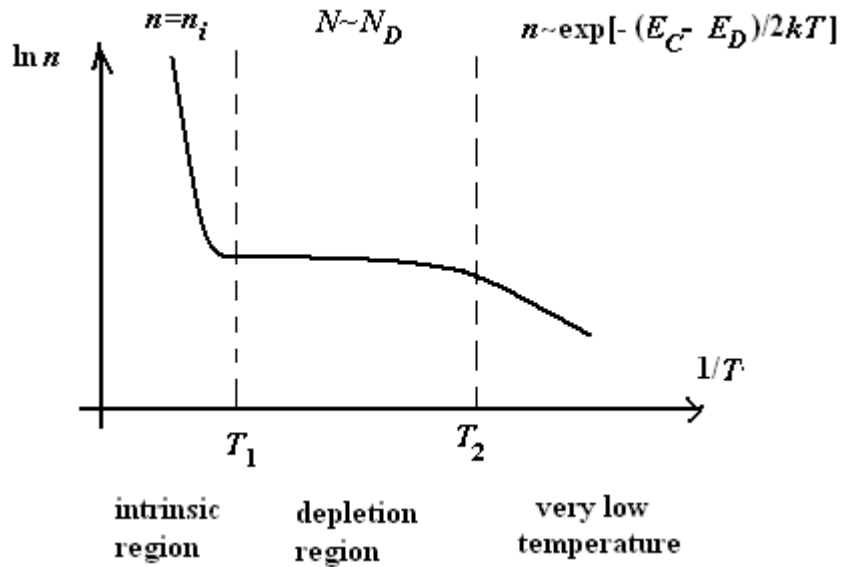
The Fermi level is given by:

$$E_F = E_C + kT \ln \left\{ \left[\frac{N_D}{2N_C} \right] \left(1 + \sqrt{1 + \frac{4N_C N_V}{N_D^2} \exp\left[-\frac{E_g}{kT}\right]} \right) \right\}$$

For high T but with $\frac{4n_i^2}{N_D^2} \ll 1$ one find again the depletion region (show!).

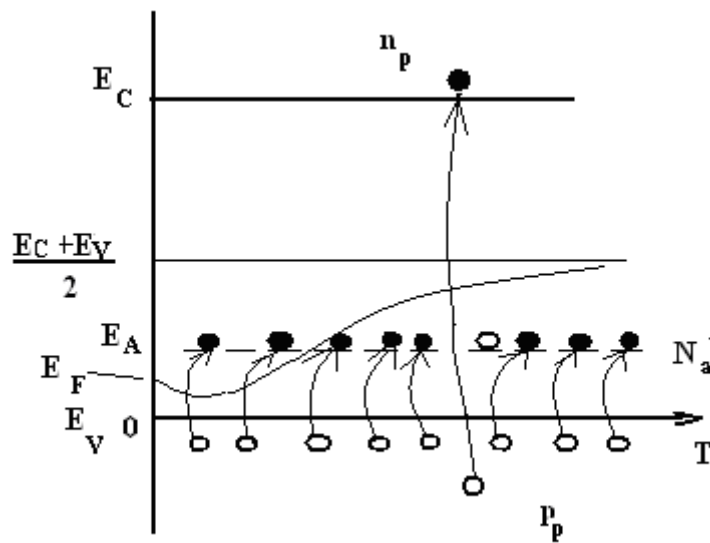
Finally, if $\frac{4n_i^2}{N_D^2} \gg 1$ the donor impurities are overtaken by the intrinsic generation

The majority carrier concentration as function of $1/T$ is given below:



b2) Type- *p* semiconductors.

Impurities from the IIIrd group e.g. *Al, Ga*. Extra levels in the gap close to E_V as in the drawing below:



N_a^- is the concentration of ionized acceptors, i.e. states filled with electrons. They obey a Fermi-Dirac statistics (up to a factor 2 in front of the exp):

$$N_a^- = \frac{N_a}{2 \exp\left[\frac{E_A - E_F}{k_B T}\right] + 1}$$

The neutrality condition is $n_p + N_a^- = p_p$. Neglecting

n_p one finds the dependence $E_F(T)$ as in the figure above. The analytic results are obtained analogous to the case of the *n*-type scnd. E.g. (QM2.55) becomes:

$$E_F = E_A - kT \left\{ \frac{1}{4} \left(-1 + \sqrt{1 + \frac{8N_A}{N_V} \exp\left[\frac{E_A - E_V}{kT}\right]} \right) \right\} \quad (\text{QM2.57})$$

E. g. for high T $N_A^- = N_A$ and $n_i \gg N_A$:

$$E_F = E_A - kT \ln \left\{ \frac{N_A}{2N_V} \left(1 + \sqrt{1 + \frac{4N_C N_V}{N_A^2} \exp\left[-\frac{E_g}{kT}\right]} \right) \right\}$$

$$p_p = \frac{N_A}{2} \left(1 + \sqrt{1 + \frac{4n_i^2}{N_A^2}} \right)$$