Fields, vectorial analysis

1. Definition and examples

When two electric charges are close enough they experience actions from each other. A similar phenomenon appears between two material bodies which attract each other. We can't say the interaction is established instantaneously (why not?). The actual model assumes that each body modifies in a specific way the region around it. We say that in the neighbor region *a field* appears.

Types of fields:

- elastic
- gravitational
- electromagnetic.

- scalar
- vectorial

You know that a positive charge Q will attract a negative one or repel another positive charge (what about the force upon itself?) denoted by q. Many of you know also the magnitude of the Coulomb force involved:

$$F = \frac{Qq}{4\pi\varepsilon r^2}\,. \qquad (1)$$

If $Qq > 0$ bodies move away, where a negative value of F implies attraction. Rather than saying that Q repels q directly, we say that Q affects the surrounding space by creating an *electric field*. The interaction is mediated by the field.

For the electric case, we define *the field strength* at a point, $\vec{E}$, as the force per unit charge that would act on a small positive test charge placed there. It is a vector, as the force is. Similar arguments hold for the gravitational field, where $\vec{g}$ characterize it.

2. Work. Line integral

If a force is constant, work is defined by $W = \vec{F} \cdot \vec{r} = FR\cos\theta$.

What if the force – or the field – varies? Elementary work is $\delta W = \vec{F} \cdot d\vec{r}$.

The total work made by the force when the body moves between points A and B is given by the *line integral* (or *path integral*):

$$W = \int_A^B \vec{F} \cdot d\vec{r} \qquad (2)$$

The integral depends on:

- the limits as in usual integral
- the integrand as in usual integral
- the actual curve between the two points **unlike in usual integrals**

How do we compute a path integral? See *Mathematical appendix "Path integrals"*. ■ (not for the exam)

3. Field lines.

One helpful way of visualizing an electric (magnetic…) field is with *field lines*. Field lines are not real, but their pattern helps us to understand interactions inside the field. The magnetic field lines of a bar-magnet are given in Fig. 1. Magnetic lines get out *North pole* and enter *South pole.*

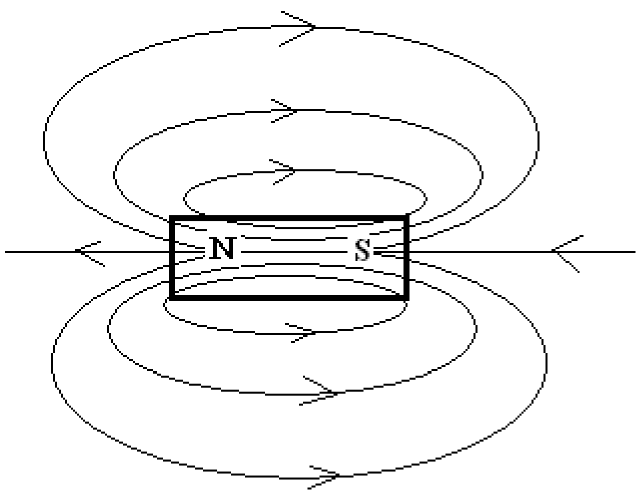
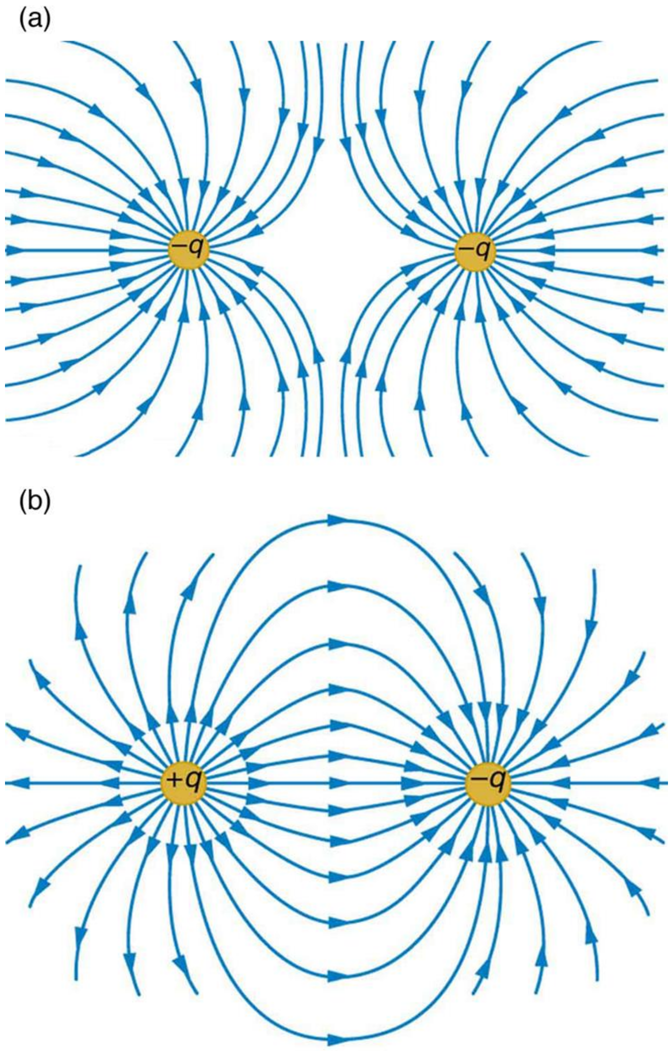


Fig. 1. Magnetic lines for a bar magnet.

Electric field lines come out + charges and enter – charges.



4. Flux. Gauss law.

Model: the electric field strength is represented by the "density" of the field lines – that is the number of lines per unit area perpendicular to the field lines

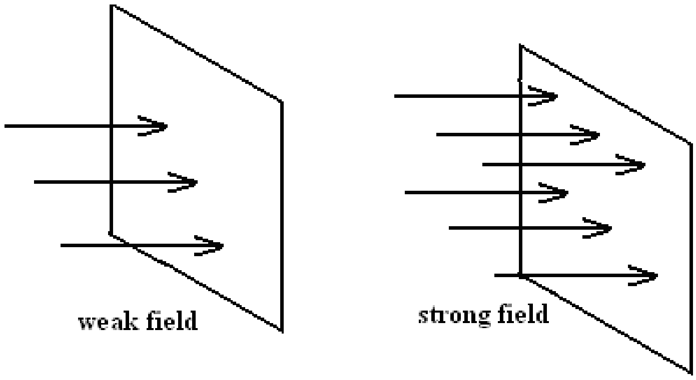


Fig.2. Field lines

$$E \propto \frac{\text{Number of field lines}}{\text{Area normal to the field}} \qquad (3)$$

The number of lines per unit area is the "density of lines". This number of field lines is in turn proportional to the charge producing the electric field,

$$\text{number of lines} \propto Q \qquad (4)$$

It turns out that the proportionality factor is ε^{-1}, where ε is the material permittivity. If charges are spread in a non-uniform way the field is not constant, neither as strength nor as direction, but the quantity $\vec{E} \cdot \mathrm{d}\vec{S}$ has the same meaning, the local number of field lines. In the situation of Fig. 3 the charge Q is surrounded by a closed surface Σ.

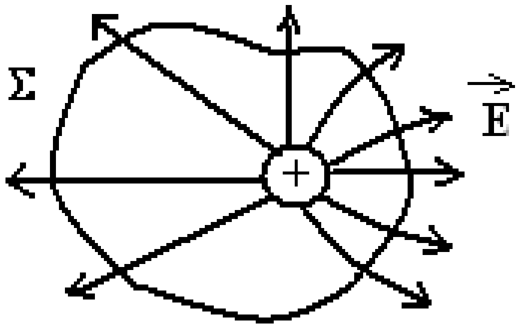


Fig.3. Electric flux through a closed surface

The *integral over the closed surface* Σ of the quantity $\vec{E}\cdot \mathrm{d}\vec{S}$ is the *electric flux*:

$$\Phi_{electric} = \oiint_{\Sigma} \vec{E}\cdot d\vec{S} \qquad (5)$$

The *Gauss law* (or the first Maxwell law) says that the electric flux equals the ratio of the total interior charge to the dielectric permittivity (ε_0 in vacuum):

$$\oiint_{\Sigma} \vec{E}\cdot d\vec{S} = \frac{Q_{total}}{\varepsilon_0} \qquad (6)$$

The integral must be thought as the limit of the sum $\sum \vec{E}\cdot\Delta\vec{S}$. Electric lines begin in the positive charges and end in the negative ones.

5. The flux law for the magnetic field

The *magnetic lines are closed lines*; there are no magnetic mono-poles. The flux law for the magnetic induction is:

$$\oiint \vec{B}\cdot d\vec{S} = 0 \qquad (7)$$

whatever the closed surface.

6. Volume integral

Spherical coordinates are r, θ and φ, as in Fig. 4 below:

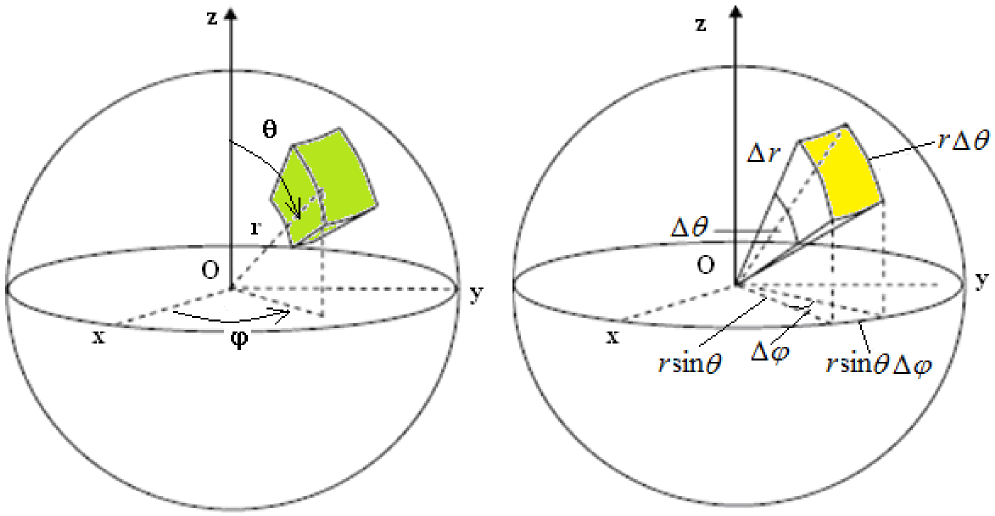


Fig. 4. Spherical coordinates (left) and spherical element of volume (right).

An element of volume of the sphere is the "spherical prism" with sides Δr, $r\Delta\theta$ and $r\sin\theta\Delta\varphi$:

$$\Delta V = r^2 \sin\theta dr d\theta d\varphi \qquad (8)$$

The volume of the sphere with radius R is:

$$V = \iiint_{sphere} r^2 \sin\theta dr d\theta d\varphi = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_0^{R} r^2 dr = \frac{4}{3}\pi R^3 \qquad (9)$$

Remark: on Fig. 4 right you may see also "the element of surface" on the sphere:

$$\mathrm{d}S = r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi \qquad (10)$$

The surface of the sphere will be:

$$S = \iint_{sphere} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi = R^2 \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi R^2 \qquad (11)$$

7. Divergence. Gauss-Ostrogradski law

The total charge of a body having a charge density defined by $\rho = \dfrac{\text{local charge}}{\text{unit of volume}}$ is given by:

$$Q = \iiint_{volume} \mathrm{d}Q = \iiint_{volume} \rho \mathrm{d}V \qquad (12)$$

The divergence of a vector $\vec{E}$ – $div\vec{E}$ – is defined as the limit of the ratio of the vector flux through a small closed surface enclosing the small volume ΔV and this volume:

$$div\vec{E} \equiv \nabla\vec{E} = \lim \frac{\oint_{\Sigma} \vec{E} \cdot d\vec{S}}{\Delta V} \qquad \text{when} \quad \Delta \mathrm{V} \to 0 \qquad (13)$$

Going to finite quantities we get the *Gauss-Ostrogradski theorem*:

$$\oint_{\Sigma} \vec{E} \cdot \mathrm{d}\vec{S} = \int_{\mathrm{V}_{\Sigma}} \nabla\vec{E} \, \mathrm{d}V \qquad (14)$$

where V_{Σ} is a volume closed by the surface Σ. From (6), (12) and (14) we get the **first Maxwell law** in local form:

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (15)$$

For the magnetic induction:

$$\nabla\vec{B} = 0 \qquad (16)$$

8. Circulation. Curl. Stokes law

The *curl* (*rotor in Romanian*) is a vector whose projection upon the normal to a surface element is the limit of the ratio between the circulation over a closed curve Γ limiting the surface ΔS and this surface:

$$\left(curl\vec{E}\right)_n \equiv \left(\nabla \times \vec{E}\right) \cdot \vec{n} = \lim \frac{\oint_\Gamma \vec{E} \cdot d\vec{r}}{\Delta S} \quad \text{when } \Delta S \to 0 \qquad (17)$$

Going to finite quantities we get the *Stokes theorem*:

$$\oint_\Gamma \vec{E} \cdot d\vec{r} = \int_{S_\Gamma} \nabla \times \vec{E} \cdot d\vec{S} \qquad (18)$$

The geometry – hat surface:

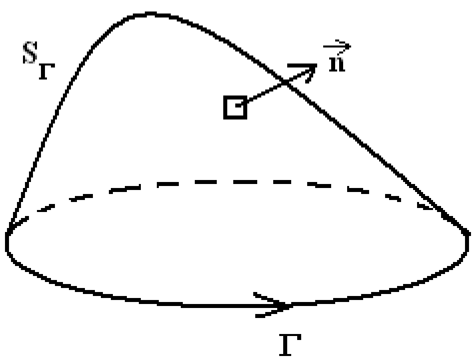


Fig. 5. Geometry of the Stokes relation