Forced damped oscillations

Equation:

$$m\ddot{x} + \gamma\dot{x} + kx = F(t) \qquad \text{(O19)}$$

or

$$\ddot{x} + \frac{2}{\tau}\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \qquad \text{(O19')}$$

Preliminary solution: Euler type because constant coefficients, but bad idea because it is an inhomogeneous equation (nonzero *RHS*).

Particular case: harmonic (one frequency, i.e. sinusoidal) driving external force.

$$F(t) = F_0 \exp[i\Omega t] \qquad \text{(O20)}$$

Solution: try

$$x(t) = a \exp[i\Omega t] \qquad \text{(O21)}$$

This is the so-called stationnary solution. We shall study this one.

Stationnary solution

(O20) and (O21) in (O19'):

$$a(\Omega) = \frac{F_0}{m} \frac{1}{\omega_0^2 - \Omega^2 + i\dfrac{2\Omega}{\tau}} = |a(\Omega)| \exp[i\theta(\Omega)]] \qquad \text{(O22)}$$

Choose: $F_0 = 1,\ m = 1,\ \omega_0 = 100$ all in SI , vary Ω and τ and represent Abs[$a(\Omega)$]= $|a(\Omega)|$ and $Arg[a(\Omega)]$.

General remarks:

- the maximum of the modulus is very close to $\Omega = \omega_0$ for small damping but may differ considerably for important losses *resonance* (see Wikipedia, the “maximum curve” from the article “Resonance”)
- examine thoroughly the parameters and the curve behavior varying the frequency of the external force.

Qualitative study.

Modulus and argument in Eq. (O22) are:

$$|a(\Omega)| = \frac{F_0}{m}\frac{1}{\sqrt{\left(\omega_0^2-\Omega^2\right)^2+\frac{4\Omega^2}{\tau^2}}} \qquad \tan\theta(\Omega) = -\frac{2\Omega/\tau}{\omega_0^2-\Omega^2} \qquad \text{(O23)}$$

External frequency very small,

$$\Omega \to 0 \quad |a(0)| = \frac{F_0}{m}\frac{1}{\omega_0^2} = \frac{F_0}{k} \qquad \theta(0) = 0 \qquad \text{(O24}'\text{)} \quad \text{(O24}''\text{)}$$

External frequency very high,

$$\Omega \to \infty \qquad |a(\infty)| \to 0 \qquad \theta(\infty) = \pi \qquad \text{(O25}'\text{)} \quad \text{(O25}''\text{)}$$

External frequency equal to the natural frequency of the oscillator,

$$\Omega = \omega_0 \quad |a(\omega_0)| = \frac{F_0}{m}\frac{1}{2\omega_0/\tau} = \frac{F_0}{m\omega_0^2}\frac{1}{2/\omega_0\tau} = Q\frac{F_0}{m\omega_0^2} = Q|a(0)| \qquad \text{(O26}'\text{)}$$

Where Q is the *quality factor of the oscillator*

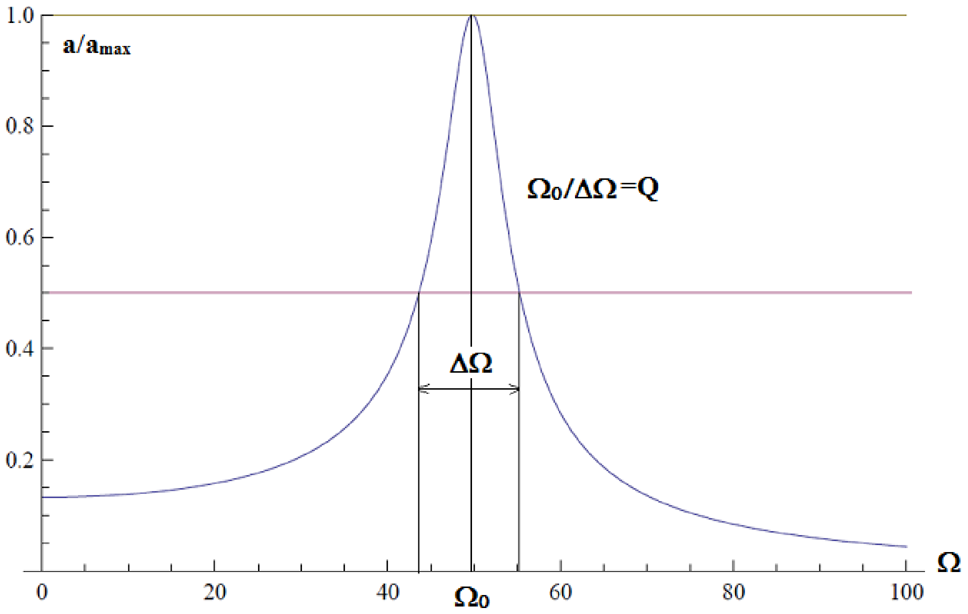
$$Q = \frac{\omega_0\tau}{2} \qquad \text{(O27)}$$

$$\theta(\omega_0) = \pi/2 \qquad \text{(O26}''\text{)}$$

The quality factor

The *quality* or the *Q-factor* is a very important quantity which controls many of the properties of an oscillating system. The maximum of the amplitude is Q times the amplitude at very low frequencies, see Eq. (O26′). The peak width is Q times narrower than the frequency for maximum amplitude. To see this let's draw a particular curve as below. We sketched the two points where the amplitude drops to half its maximum value. The difference between the two pulsations is known as *the full width at half-maximum* (FWHM) $\Delta\Omega$. It must be shown that a very good approximation gives the ratio of the central frequency to the FWHM as

$$\frac{\omega_0}{\Delta\Omega} \cong Q \qquad \text{(O28)}$$



The maximum is not exactly in $\Omega = \omega_0$. A little calculus shows that the maximum appears for

$$\Omega_{max} = \omega_0\sqrt{1-\frac{1}{2Q^2}} \qquad \text{(O29)}$$

Exercise: compute the value of the maximum amplitude and write it in the form

$$|a(\Omega_{max})| = |a(\omega_0)|\, f_1(Q) \approx |a(\omega_0)|(1+f_2(Q))$$

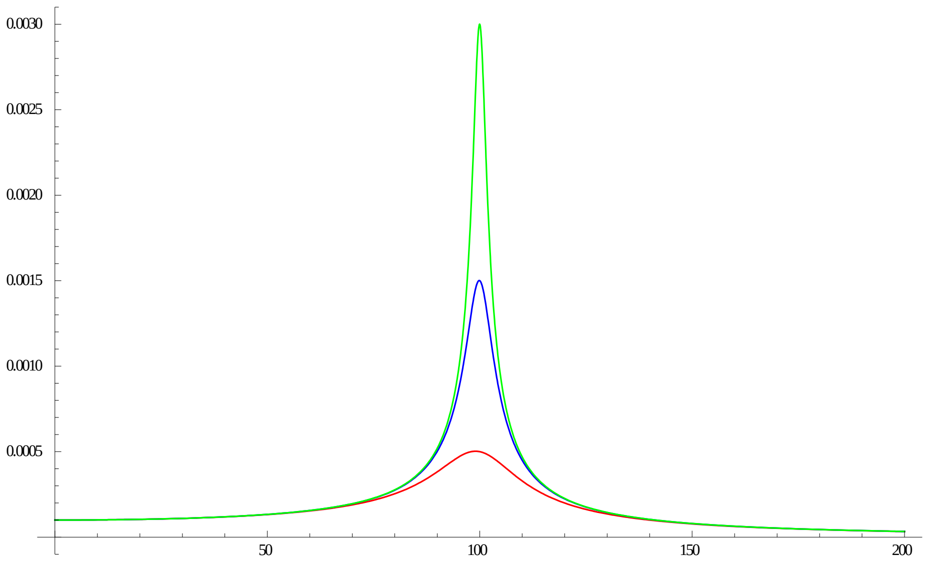
For details see **http://www.math.dartmouth.edu/~ahb/scia49/q.pdf** for a quick view of the Q factor. See also **http://tf.nist.gov/general/enc-q.htm** (**National Institute of Standards and Technology**). ...The quality factor or *Q* factor is...

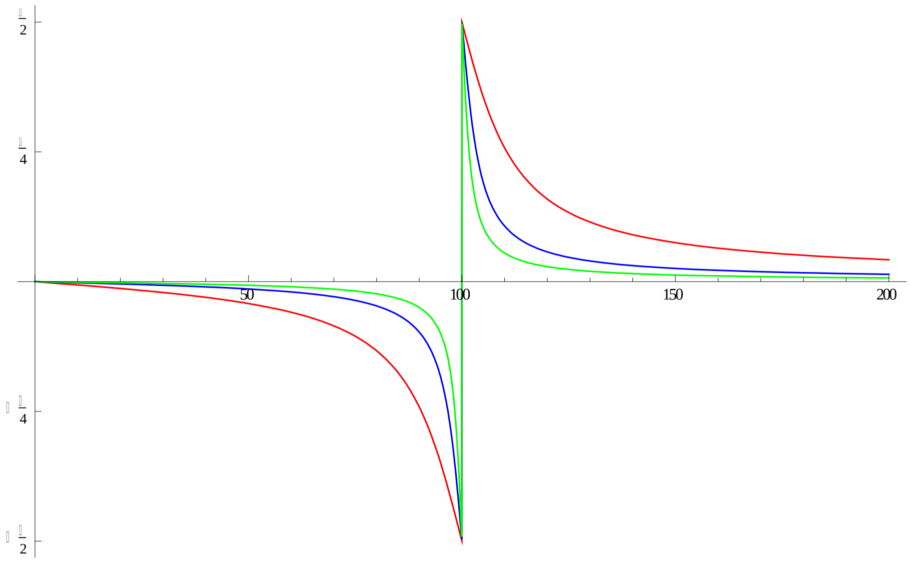
> an inherent characteristic of an oscillator that influences its stability. The quality factor, Q, of an oscillator is defined as its resonance frequency divided by its resonance width. Obviously a high resonance frequency and a narrow resonance width are both advantages when seeking a high Q. Generally speaking, the higher the Q, the more stable the oscillator, since a high Q means that an oscillator will stay close to its natural resonance frequency. The

table shows some approximate Q values for several different types of oscillators.

Oscillator Type	Quality Factor, Q
Tuning Fork	10^3
Quartz Wristwatch	10^4
OCXO	10^6
Rubidium atom	10^7
Cesium Beam	10^8
Hydrogen Maser	10^9
Cesium Fountain	10^{10}
Mercury Ion Optical Standard	10^{14}
Mechanical oscillators	1…10
RLC oscillators	10…100

```
ClearAll[a,amod,aarg,om,tau,om0,f0,m,q,theta]
om0=100;f0=1;m=1;
amod[om_,tau_]=f0/(m*Sqrt[(om0^2-om^2)^2+4*om^2/tau^2])
aarg[om_,tau_]=ArcTan[-(2*om/tau)/(om0^2-om^2)]
Plot[{amod[om,0.1],amod[om,0.3],amod[om,0.6]},{om,0,200},
PlotRange→All,PlotStyle→{Red,Blue,Green}]
Plot[{aarg[om,0.1],aarg[om,0.3],aarg[om,0.6]},{om,0,200},Ticks→
{Automatic,{0,Pi/2,-Pi/2,Pi/4,-Pi/4}},PlotRange→All,PlotStyle→{Red,Blue,Green}]
```





amod[om0,0.1]
FindMaximum[Abs[a[om,0.1]],{om,om0}]
0.0005
{0.000502519,{om→98.9949}}

amod[om0,0.3]
FindMaximum[Abs[a[om,0.3]],{om,om0}]
0.0015
{0.00150083,{om→99.8888}}
amod[om0,0.6]
FindMaximum[Abs[a[om,0.6]],{om,om0}]
0.003
{0.00300042,{om→99.9722}}

This stationnary solution is correct only after a certain amount of time, when the oscillator „forgets" initial conditions and its natural frequency ω_0. The complete solution contains also an initial behavior, called transient solution.

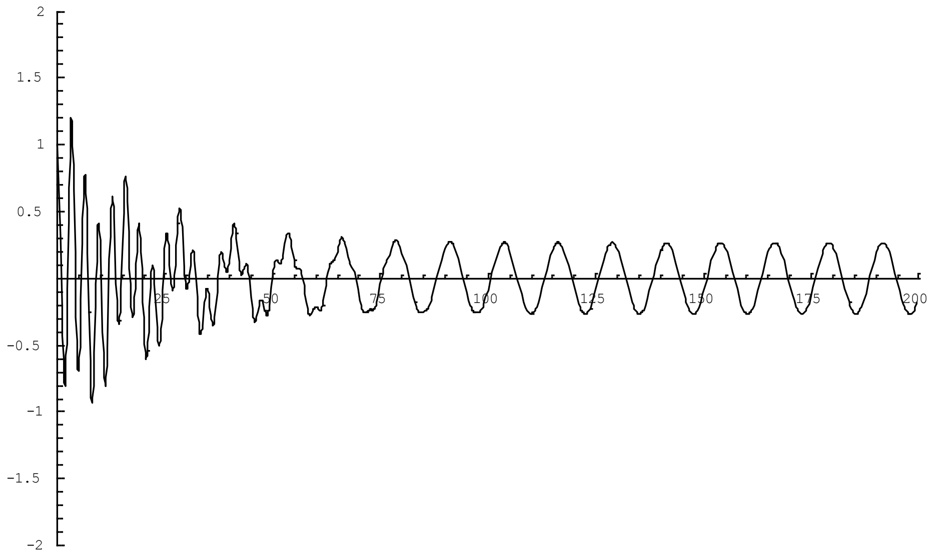
Transient solution

Mathematics shows that the general solution of an inhomogeneous equation as

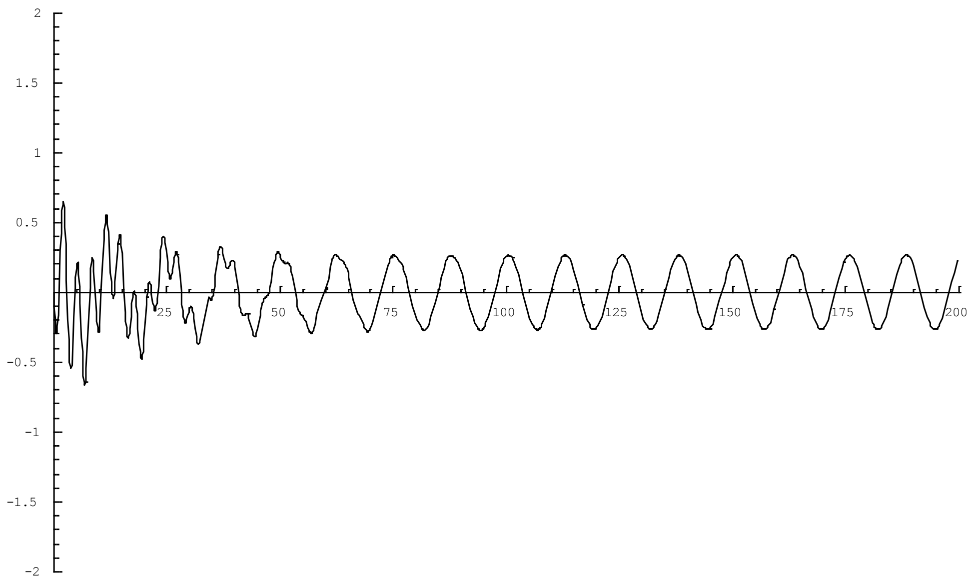
$$\ddot{x} + \frac{2}{\tau}\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \qquad \text{(O19')}$$

contains the general solution af the homogeneous equation as e.g. (O15) added with a particular solution of the complete equation, as e.g. (O22). The following programs show several possibilities; in all situations the permanent (stationnary) solution appears after a certain time. The two regimes are clerly separated.

```
ClearAll[A,x,t,tau,om1,om2,phi,y]
tau=10;om1=2;om2=0.5;phi=0;A=1;
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],x[0]□1,x'[0]
□1},x[t],t]
Plot[Re[x[t]/.sol],{t,0,2000},PlotRange→{{0,200},{-
2,2}},PlotPoints→500]
```

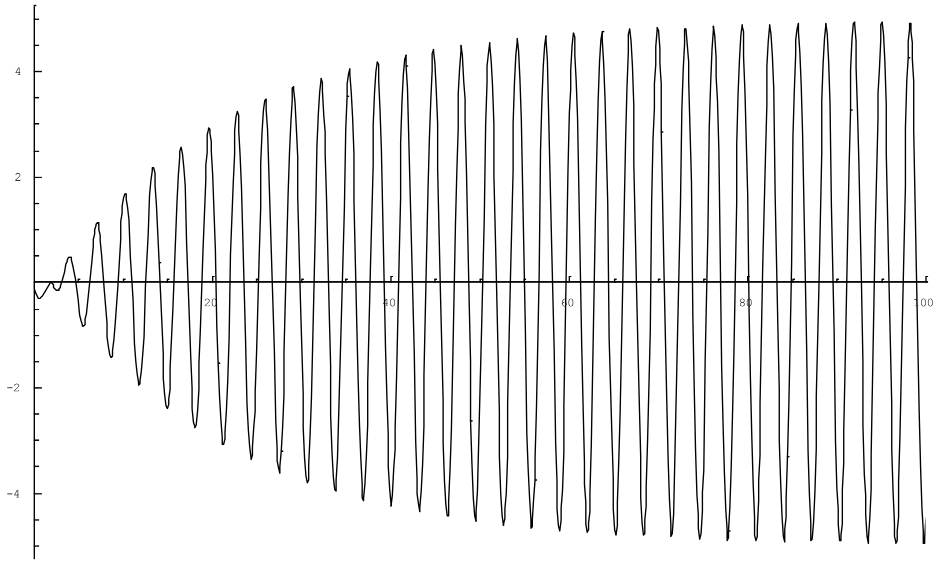



```
ClearAll[A,x,t,tau,om1,om2,phi,y]
tau=10;om1=2;om2=0.5;phi=Pi/2;A=1;
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],x[0]□0,x'[0]
□-1},x[t],t];
Plot[Re[x[t]/.sol],{t,0,2000},PlotRange→{{0,200},{-
2,2}},PlotPoints→500]
```

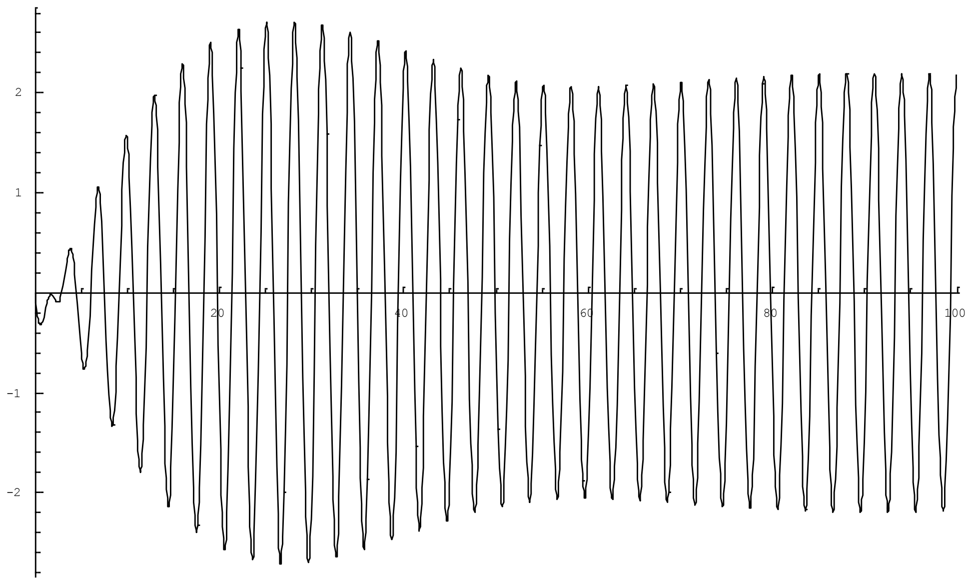



```
ClearAll[A,x,t,tau,om1,om2,phi,y]
tau=10;om1=2;om2=2;phi=Pi/2;A=1;
```

```
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],x[0]□0,x'[0]
□-1},x[t],t];
Plot[Re[x[t]/.sol],{t,0,2000},PlotRange→{{0,100},All},PlotPoints→500
]
```

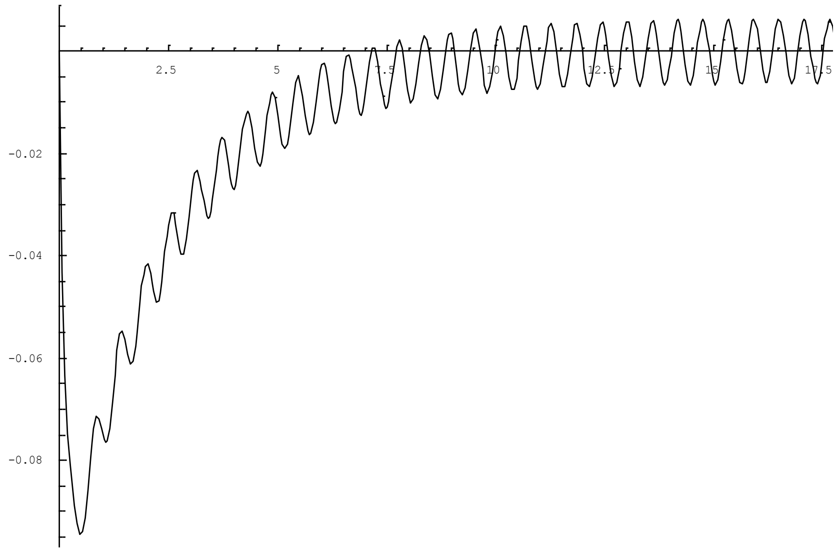



```
ClearAll[A,x,t,tau,om1,om2,phi,y]
tau=10;om1=2;om2=2.1;phi=Pi/2;A=1;
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],x[0]□0,x'[0]
□-1},x[t],t];
Plot[Re[x[t]/.sol],{t,0,2000},PlotRange→{{0,100},All},PlotPoints→10
00]
```

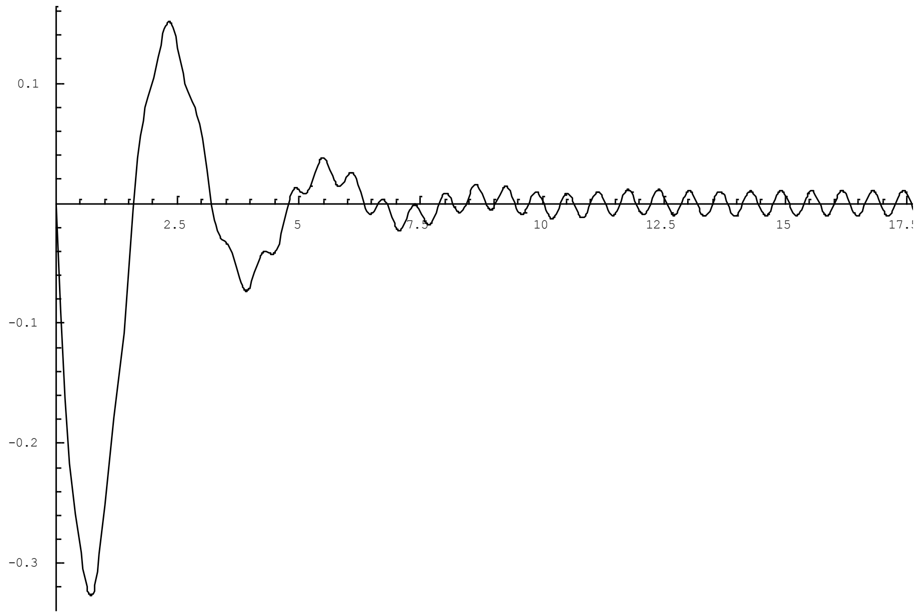



```
ClearAll[A,x,t,tau,om1,om2,phi,y]
tau=0.1;om1=2;om2=10.8;phi=Pi/2;A=1;
```

```
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],x[0]□0,x'[0]□-
1},x[t],t];
Plot[Re[x[t]/.sol],{t,0,2000},PlotRange→{{0,20},All},PlotPoints→1000]
```

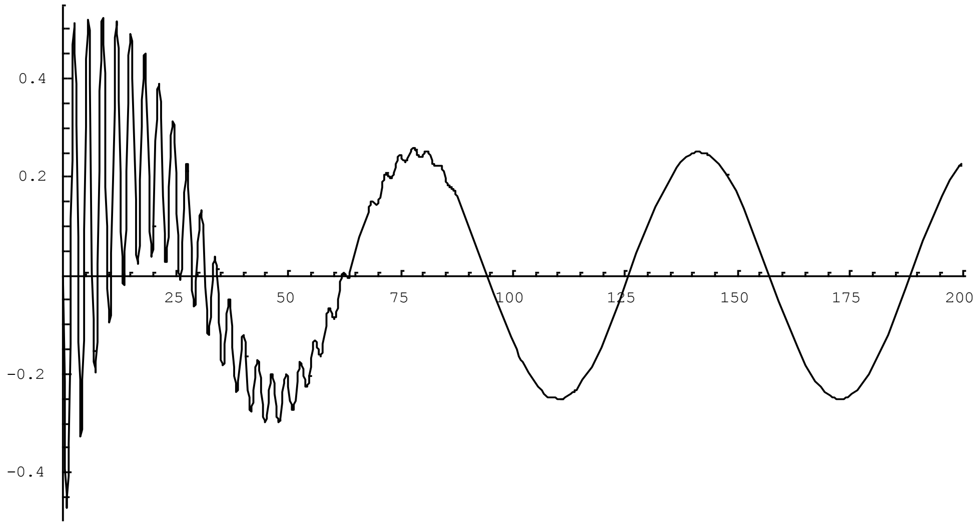



```
ClearAll[A,x,t,tau,om1,om2,phi,y]
tau=1;om1=2;om2=10;phi=0;A=1;
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],x[0]□0,x'[0]
□-1},x[t],t];
Plot[Re[x[t]/.sol],{t,0,200},PlotRange→{{0,20},All},PlotPoints→1000
]
```



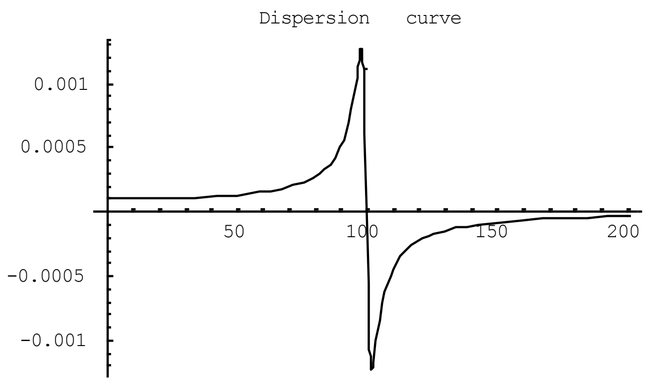

```
ClearAll[A,x,t,tau,om1,om2,phi,y]
tau=10;om1=2;om2=0.1;phi=0;A=1;
```

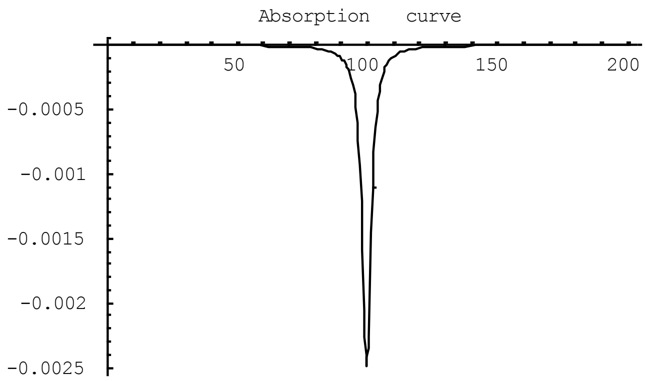
```
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],x[0]□0,x'[0]□-1},x[t],t];
Plot[Re[x[t]/.sol],{t,0,2000},PlotRange→{{0,200},All},PlotPoints→500]
```




```
ClearAll[a,om,tau,om0,f0]
om0=100;f0=1;m=1;tau=0.5;
a[om_,tau_]=f0/(m*(om0^2-om^2+I*2*om/tau))
Plot[Re[a[om,1]],{om,0,200},PlotRange→All,PlotLabel->"Dispersion curve"]
Plot[Im[a[om,1]],{om,0,200},PlotRange→All,PlotLabel->"Absorption curve"]
```

$$\frac{1}{10000 \square 4.\ \square\ om\ \square om^2}$$

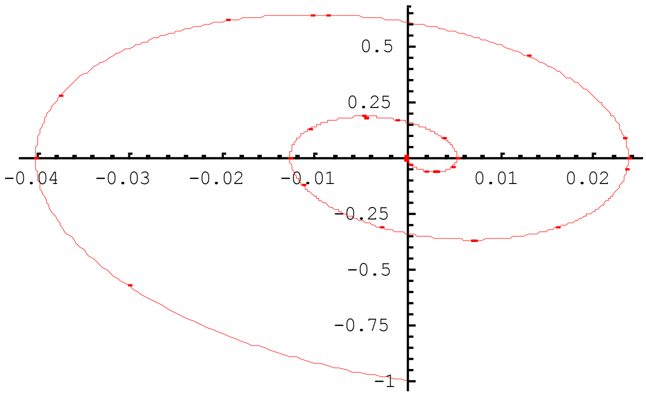



Absorption curve
50
100
150
200
-0.0005
-0.001
-0.0015
-0.002
-0.0025

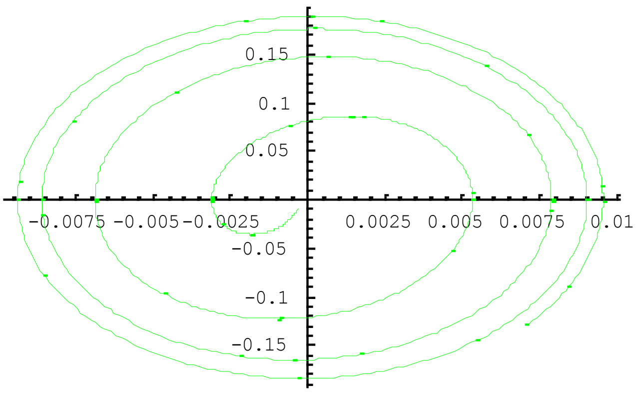
Phase space figures. Forced damped oscillator. Resonance, om1=om2=20, phase shift of Pi/2, tau=0.2 (all in SI).

```
ClearAll[A,x,t,tau,om1,om2,phi,y,sol,xsolfor]
tau=0.2;om1=20;om2=20.0;phi=Pi/2;A=1;
sol=DSolve[{x''[t]+x'[t]/tau+om1^2*x[t]□A*Sin[om2*t+phi],
x[0]□0,x'[0]□-1},x[t],t]
```

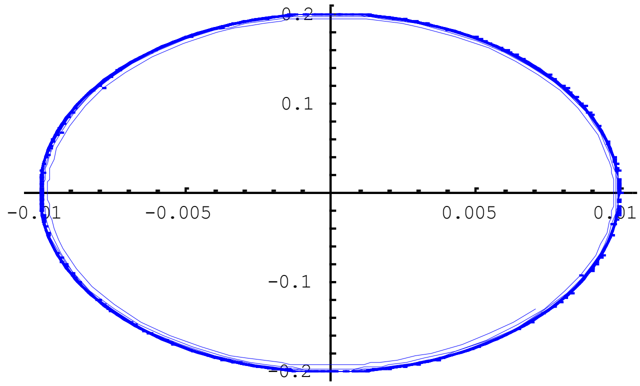
In red : $t \in (0,\ 0.75)$, the damping exceeds the influence of the external force, the point goes from the initial phase $x(0) = 0,\ \dot{x}(0) = -1$ to the origin $x(0) = 0,\ \dot{x}(0) = 0$.



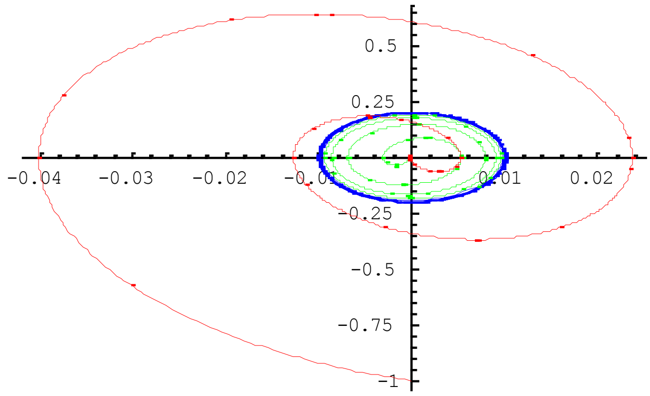
In green : $t \in (0.75,\ 2)$, the external force re-drives the point which moves according to a superposition of the driving force and the free damped oscillation.



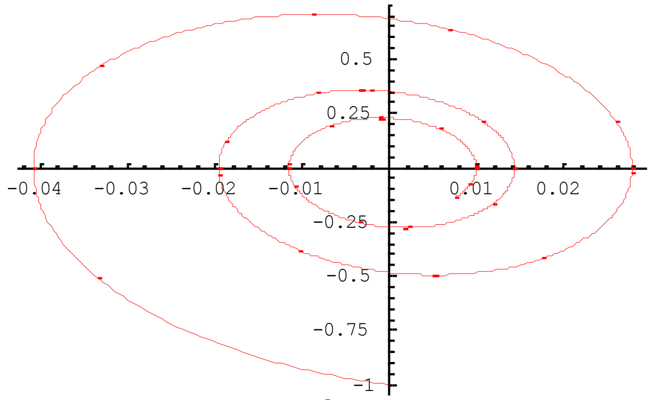
In blue $t \in (2, 50)$ the point moves in the stationary part,

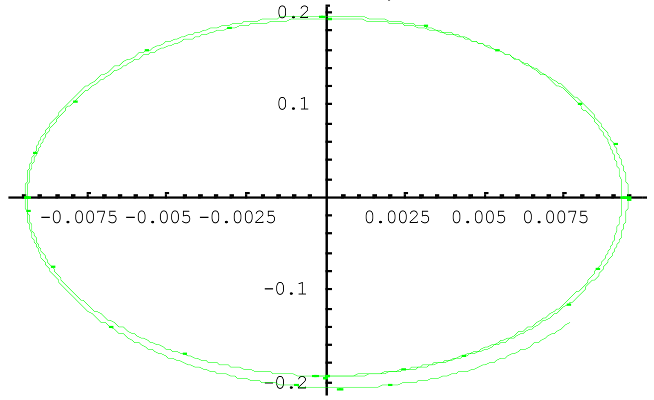


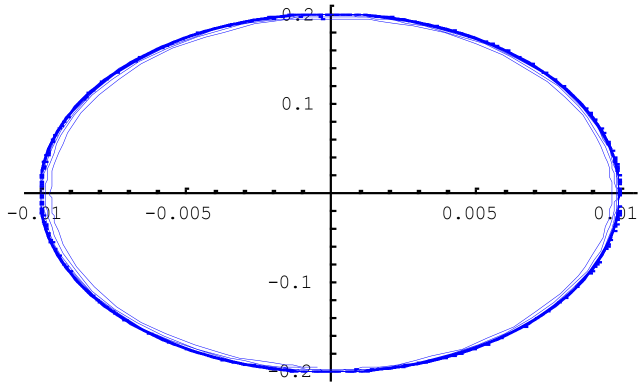
All the three regions put together.

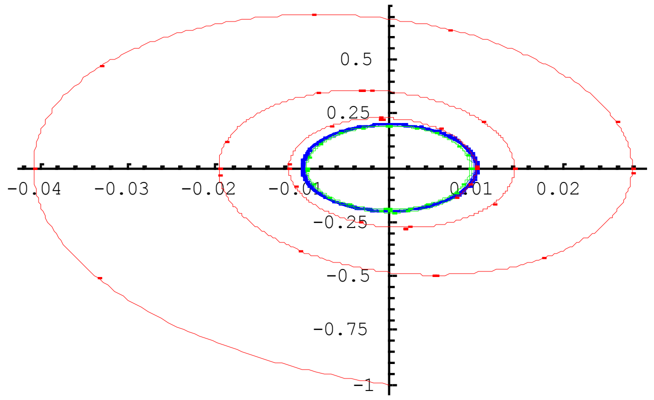


The same parameters, but no phase shift. The oscillator no more arrives in the origin.









No more resonance : om1=20, om2=22. The final amplitude is smaller than before.