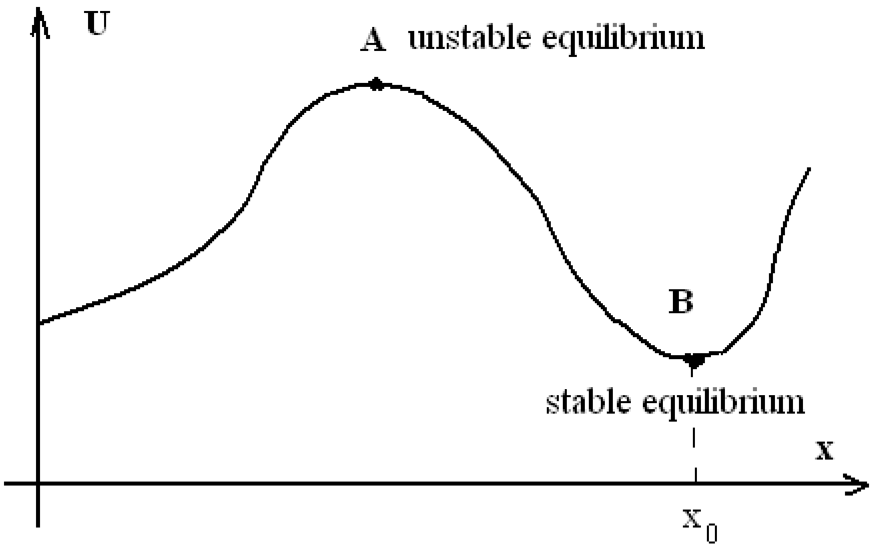
Oscillations

O1. 1. Free 1D oscillations (harmonic oscillations)

Newtonian mechanics applies the 2nd law and solves DE given initial conditions for positions and velocities. Theorems of variation and conservation facilitate and speed up the process. In 1D case the potential energy varies as in the figure:



In points *A* and *B* the system is in equilibrium because $\frac{dU}{dx}=0$ hence the force is zero. *Question*: why *A* is stable and *B* unstable?

In a small neighborhood of *B* the potential may be expanded in a Taylor series and we may keep up only the first terms:

$$U(x)\cong U(x_0)+\frac{dU}{dx}\Big|_{x_0}(x-x_0)+\frac{1}{2}\frac{d^2U}{dx^2}\Big|_{x_0}(x-x_0)^2 =$$

$$U(x_0)+\frac{1}{2}\frac{d^2U}{dx^2}\Big|_{x_0}(x-x_0)^2$$

because $\frac{dU}{dx}\Big|_{x_0}=0$ in point *B*. The force in the vicinity of *B* is $F=-\frac{dU}{dx}\cong-\frac{d^2U}{dx^2}\Big|_{x_0}(x-x_0)=-k(x-x_0)$ with k>0 the positive value of the second derivative in x_0. **The elastic force is a first approximation – the linear approximation –** of the force around an equilibrium point of the system.

Newton's 2nd law: $m\ddot{x} = -kx$. Notation: $\sqrt{\frac{k}{m}} = \omega_0$.

$$\ddot{x} + \omega_0^2 x = 0 \tag{O1}$$

Assume $\quad x(0) = x_0, \quad \dot{x}(0) = v_0$

Search solutions of the form (Euler): $x(t) = A\exp[rt]$. One finds $r_{1,2} = \pm i\omega_0$ hence $x(t) = A\exp[i\omega_0 t] + B\exp[-i\omega_0 t]$. Initial conditions:

$A + B = x_0$, $i\omega_0(A - B) = v_0$, $\quad A = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega_0}\right)$, $\quad B = \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega_0}\right)$ and

the final solution writes:

$$\begin{aligned} x(t) &= \frac{1}{2}x_0\left(\exp[i\omega_0 t] + \exp[-i\omega_0 t]\right) + \frac{1}{2}\frac{v_0}{i\omega_0}\left(\exp[i\omega_0 t] - \exp[-i\omega_0 t]\right) = \\ &= x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t) = C\sin(\omega_0 t + \varphi_0) \end{aligned} \tag{O2}$$

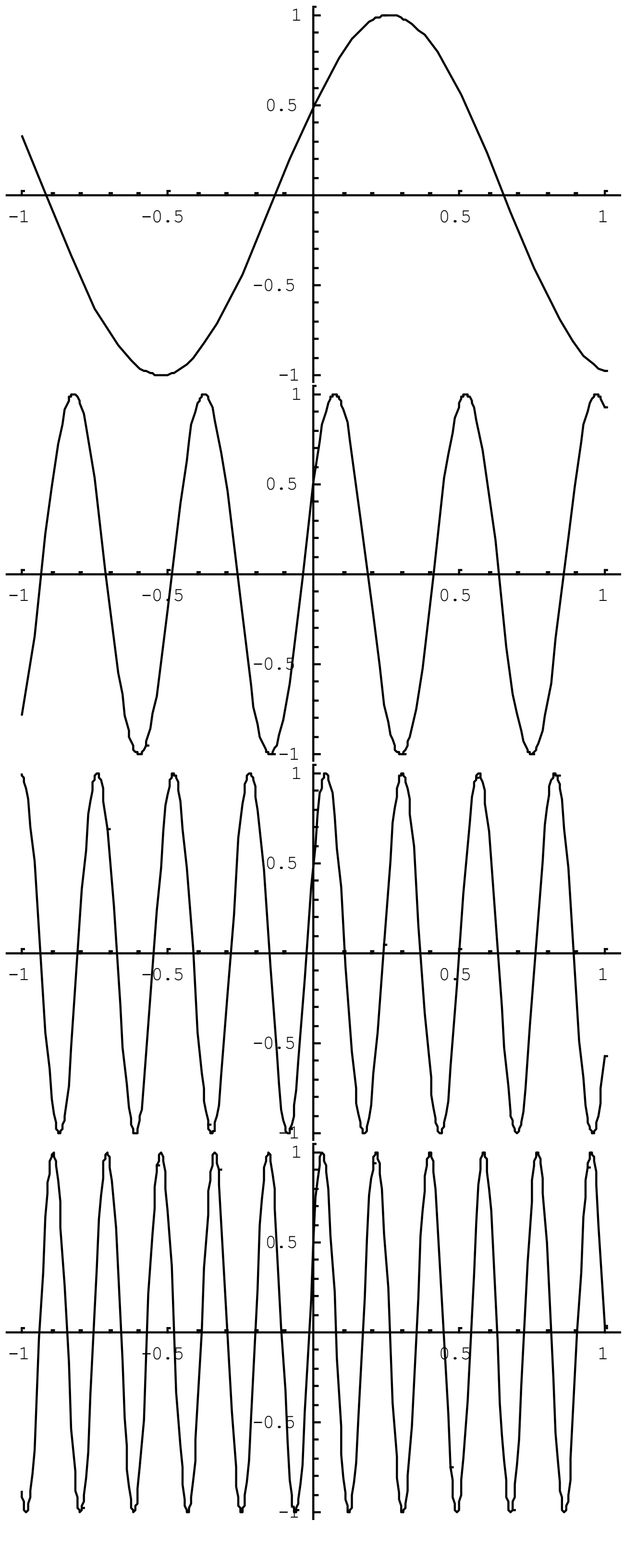
1. Verify that the initial conditions are satisfied.
2. Compute C and φ_0.
3. Search from the beginning for a solution of the type $x(t) = C\sin(\omega_0 t + \varphi_0)$
4. Draw the solution for C=1 cm, $\omega_0 = 4\,\text{s}^{-1}$, $\varphi_o = \pi/6$, and for $-1\text{s} \le t \le 1\text{s}$.
5. Compute the kinetic energy and the potential energy if m=100 g. Draw the graph for E_k for U and for the total energy in the interval $-1\text{s} \le t \le 1\text{s}$.

The following graphics show different views of the function $x(t) = C\sin(\omega_0 t + \varphi_0)$ and of the kinetic, potential and total energies for various ω_0 and φ_0 together with the programming lines in Mathematica which allow drawings.

```
ClearAll[x,t,c,om,phi,a,b,ek,u,m]
m=1;
x[c_,t_,om_,phi_]=c*Sin[om*t+phi];
ek[c_,t_,om_,phi_]=m*(D[x[c,t,om,phi],t])^2/2
u[c_,t_,om_,phi_]=m*(om*x[c,t,om,phi])^2/2
```

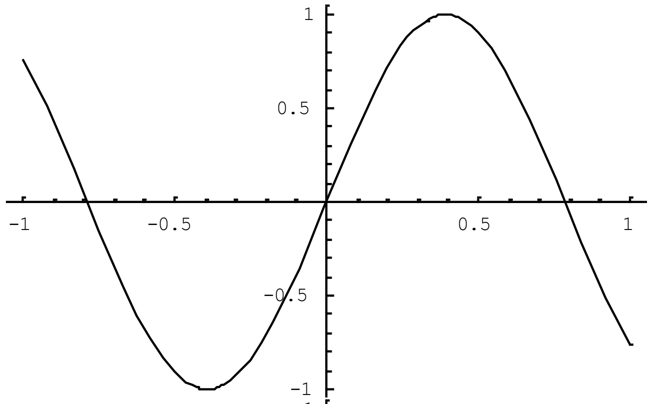
Different frequencies, from 4 s^{-1} to 40 s^{-1}, in steps of 10 s^{-1}

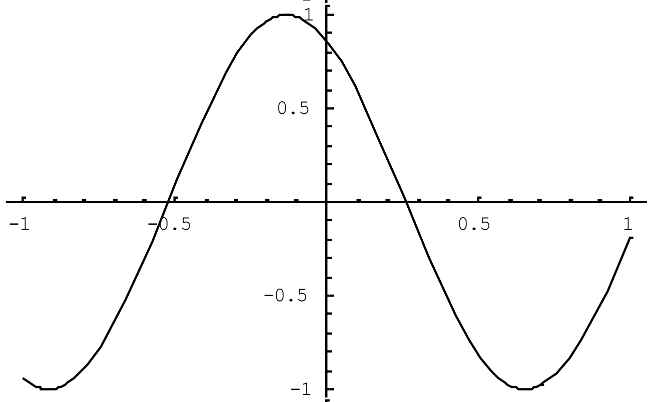
```
Do[Plot[x[1,t,n,Pi/6],{t,-1,1}],{n,4,40,10}]
```

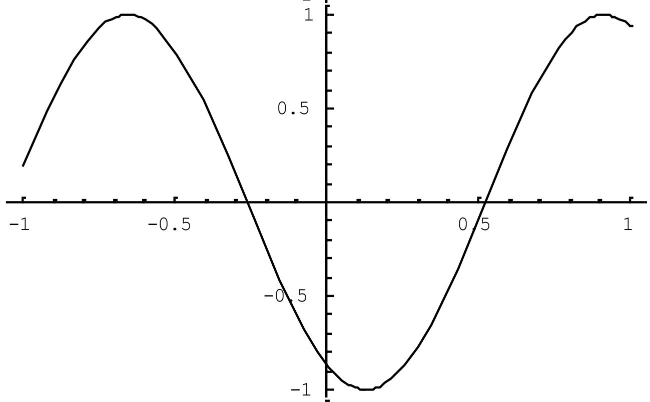


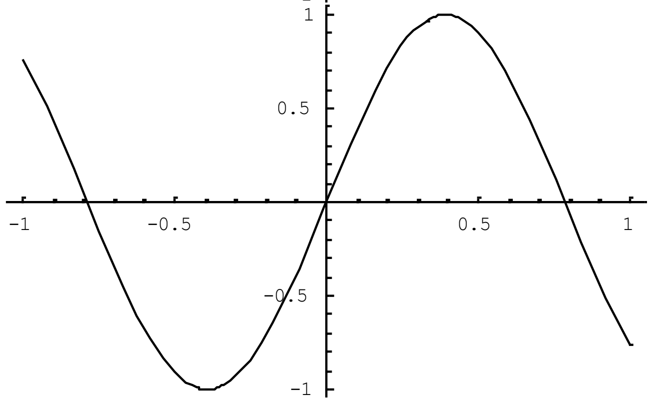
Different initial phases, from 0 to 2Pi, in steps of 2Pi/3

```
Do[Plot[x[1,t,4,n*Pi/6],{t,-1,1}],{n,0,12,4}]
```



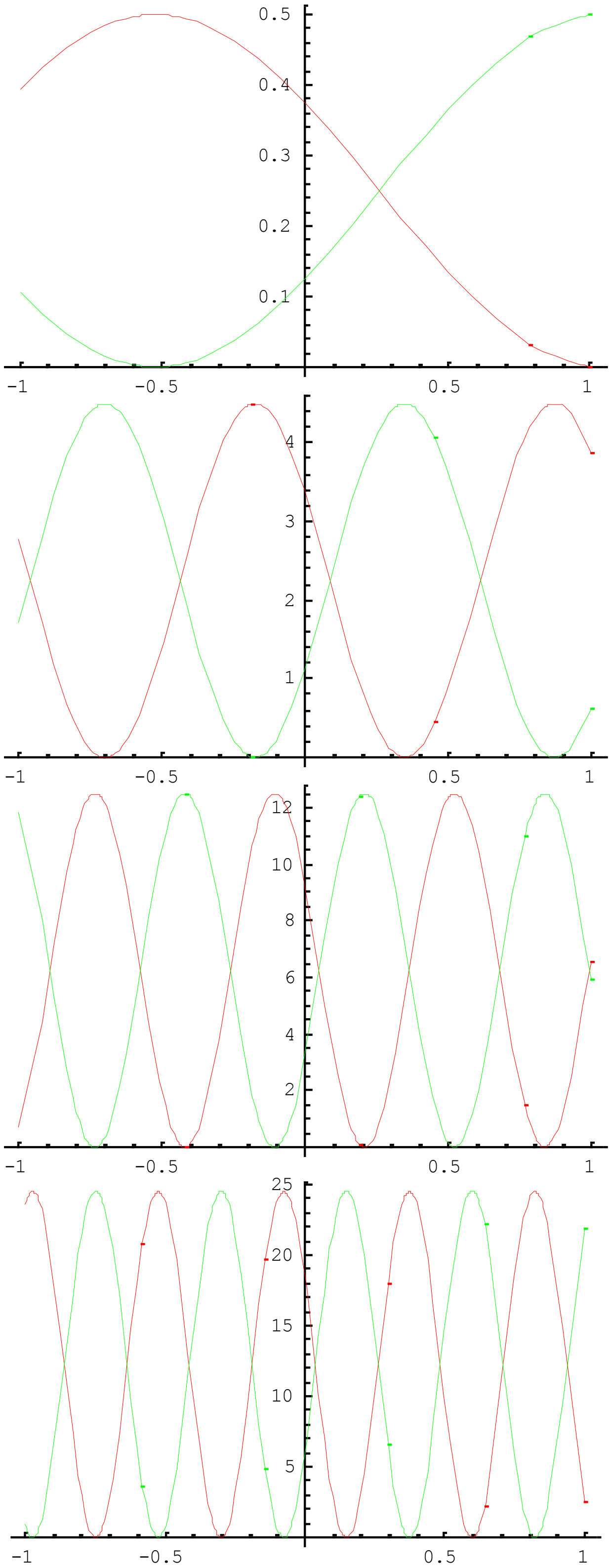


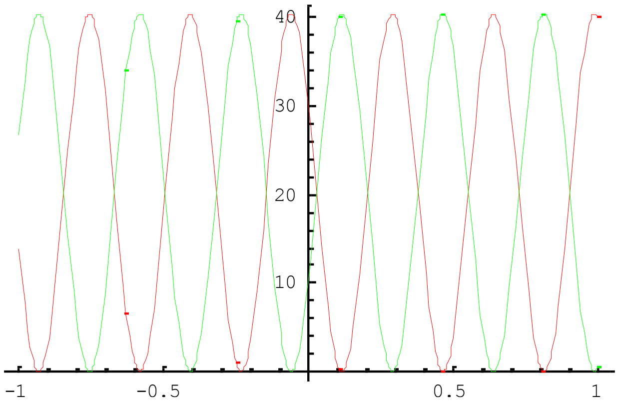




Kinetic and potential energies for frequencies between 1 $s^{-1}$and 9 s^{-1} in steps of 2 s^{-1}.

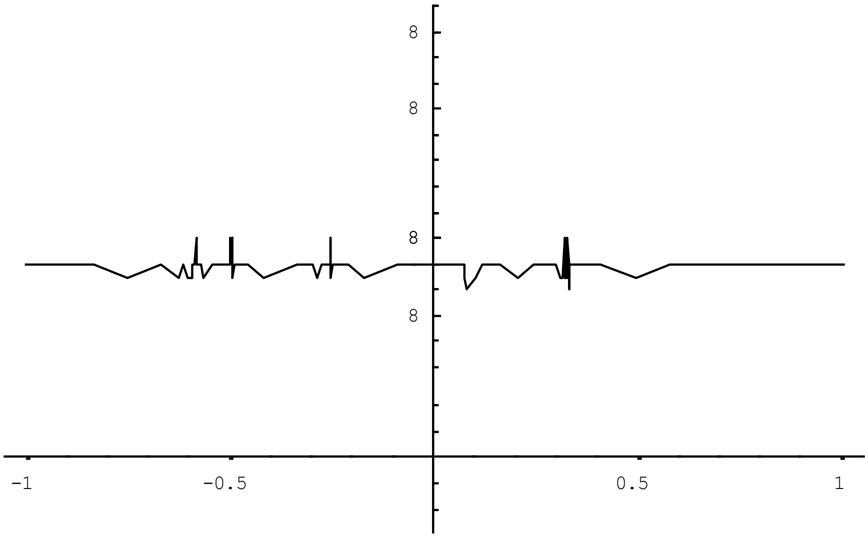
```
Do[Plot[{ek[1,t,n,Pi/6],u[1,t,n,Pi/6]},{t,-
1,1},PlotStyle→{RGBColor[1,0,0],RGBColor[0,1,0]}],{n,1,9,
2}]
```





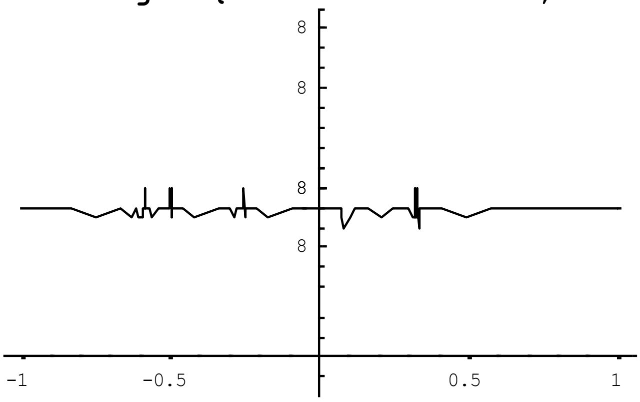
Sum of potential and kinetic energies for $\omega_0 = 4\,s^{-1}$.

```
Plot[u[1,t,4,Pi/6]+ek[1,t,4,Pi/6],{t,-1,1}]
```



Is it not a constant ??? Well, almost, with around $10^{-14}/8$ relative errors, see below.

```
Plot[u[1,t,4,Pi/6]+ek[1,t,4,Pi/6],{t,-1,1},
PlotRange->{7.99999999999999,8.00000000000001}]
```



O1. 2. Free 1D oscillations. Phase space.

The phase space Γ is an abstract space; for one material point in 1D it has two coordinates: the usual Cartesian coordinate x and the linear momentum p. For n material points in 3D it has $6n$ coordinates, the $3n$ coordinates of all the points and the $3n$ components of their moments.

Sometimes one uses coordinates and velocities. For just one point moving in 1D the coordinates are $(x, \dot{x})$. The trajectory of this point during a free oscillation is given in parametrical form by:

$$\begin{cases} x(t) = C\sin(\omega_0 t + \varphi_0) \\ \dot{x}(t) = C\omega_0 \cos(\omega_0 t + \varphi_0) \end{cases} \tag{O3}$$

This is an ellipsis along which the point moves over and over.