Mathematical appendix 2

Partial derivatives

(Ken Riley, Matthew Hobson, *Mathematical methods for physics and engineering*, Cambridge 2002)

Partial differentiation

Let $f_1(x)=\cos^2 x$. Consider functions that depend on more than one variable, e.g. function $g(x,\ y)=x^2+3xy$, or $f(x,y)=\cos^2 x+\cos^2 y$, or $h(x_1,x_2,...,x_n)=x_1^2+x_2^2+...x_n^2$.

Plots of $f_1(x)$ and $g(x,y)$:

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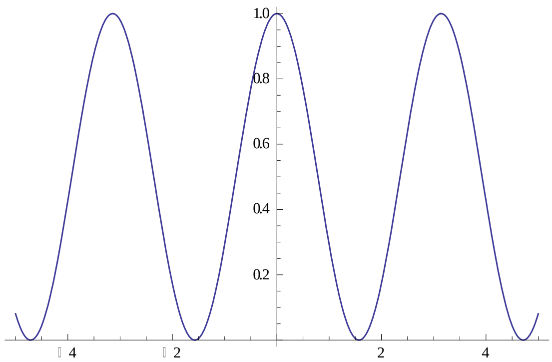
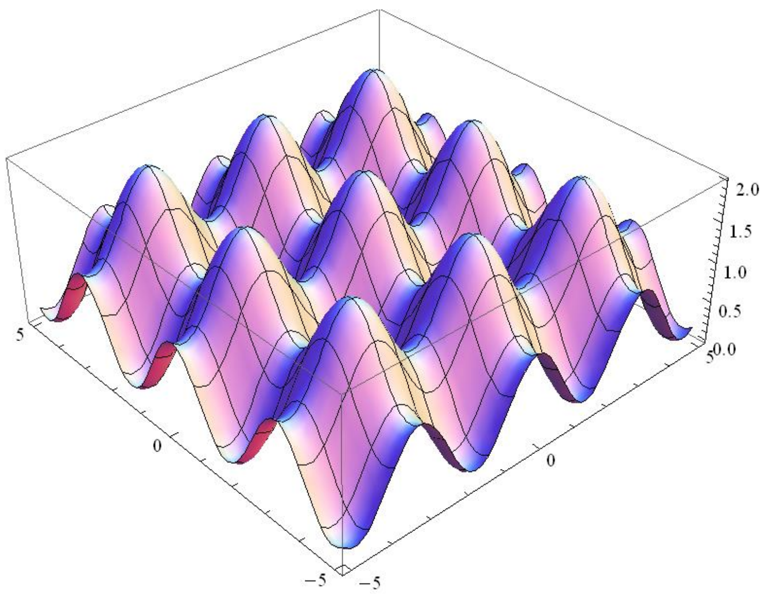
ClearAll[x,y,f1,g]
f1[x_]=(Cos[x])^2
f[x_,y_]=(Cos[x])^2+(Cos[y])^2
Plot[f1[x],{x,-5,5}]
Plot3D[f[x,y],{x,0-5,5},{y,-5,5}]
Cos[x]^2
Cos[x]^2+Cos[y]^2
```





It is clear that a function of two variables will have a gradient in all directions in the xy-plane. The rate of change of $f(x, y)$ in the positive x- and$y$-directions are the *partial derivatives* with respect to x and y respectively. Definitions (if limits exist):

$$\frac{\partial f}{\partial x} = \lim \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \quad \text{for} \quad \Delta x \to 0$$

$$\frac{\partial f}{\partial y} = \lim \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \quad \text{for} \quad \Delta y \to 0$$

and in general:

$$\frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_i} = \lim \frac{f(x_1, x_2, ..., x_i + \Delta x_i, ..., x_n) - f(x_1, x_2, ..., x_i, ..., x_n)}{\Delta x_i} \quad \text{for} \quad \Delta x_i \to 0$$

Higher derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

An important theorem states that the second partial derivatives are equal, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, provided they are continuous. Examples.

Total differential and total derivative

We compute the rate of change of $f(x, y)$ in an arbitrary direction. We make simultaneous small changes in x and Δy in y. As a result, f changes to $f + \Delta f$ and we may write:

$$\Delta f = f(x+\Delta x,\ y+\Delta y) - f(x,\ y) =$$
$$f(x+\Delta x,\ y+\Delta y) - f(x,\ y+\Delta y) + f(x,\ y+\Delta y) - f(x,\ y) =$$
$$\frac{f(x+\Delta x,\ y+\Delta y) - f(x,\ y+\Delta y)}{\Delta x}\Delta x + \frac{f(x,\ y+\Delta y) - f(x,\ y)}{\Delta y}\Delta y$$

For not too large Δx and Δy we may write:

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

When Δx and Δy become infinitesimal define *the total differential df* of the function *f*(*x*, *y*):

$$\mathrm{d}f \approx \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

Exact and inexact differentials

We know how to compute total differentials when we know the functions. Sometimes we want to revrse the process and find the function *f* that differentiates to give a known differential. Examples: $xdy + ydx$ is the differential of *f*(*x*, *y*)=*xy*+*c*; $xdx + ydy$ is the differential of ...

On the other hand, the differential $xdy + 3ydx$ is inexact. Indeed, assume $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = xdy + 3ydx$, hence $\frac{\partial f}{\partial x} = 3y$, $\frac{\partial f}{\partial y} = x$. The mixed second order derivatives are not equal therefore there is no such a function with $xdy + 3ydx$ as an exact differential. But we may find an *integrating factor* which, transforms the given inexact differential in an exact one. In our case this is x^2:

$$x^2 \times (xdy + 3ydx) = 3x^2ydx + x^3dy = d(x^3y)$$