9. Interference

1. General frame

We call *interference* the superposition of two or several waves, in particular em waves. The materials are assumed to be linear, so we can apply the superposition principle.

Interference appears under certain conditions when two or more waves superpose. We study only interference of em waves, in particular optical waves.

Question: what is light ?

When light from two sources superposes what we observe usually is an increase of the overall intensity. If the two sources are *coherent*, one observes a totally different spatial pattern formed by maxima and minima which remain stable during the observation. This is the so-called *stationary interference*. We shall deal with with this type of superposition; it is the foundation of interferometry. Interferometry with elmgn waves is one of the most accurate measurement methods in physics.

Assume that in a region superpose two em waves. The total electric field is:

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 \qquad \text{(I1)}$$

The total intensity is:

$$I_{tot} = \frac{1}{2\zeta}\left|\vec{E}_{tot}\right|^2 \propto \left(\vec{E}_1 + \vec{E}_2\right)\left(\vec{E}^*{}_1 + \vec{E}^*{}_2\right) = I_1 + I_2 + \vec{E}_1 \cdot \vec{E}_2^* + \vec{E}_2 \cdot \vec{E}_1^* \qquad \text{(I2)}$$

The interference term is $I_{12} = \vec{E}_1 \cdot \vec{E}_2^* + \vec{E}_2 \cdot \vec{E}_1^*$, hence

$$I_{tot} = I_1 + I_2 + I_{12} \qquad \text{(I3)}$$

When $I_{12} \neq 0$ we have *constructive interference*, giving interference *fringes*. One says the two waves are *coherent*.

2. Coherence conditions

In real conditions one does not detect instantaneous intensities, because visible light has around 10^{14}-10^{15} Hz, so we measure an average made on 10^{5}-10^{13} periods. Relation (I3) has to be written:

$$\langle I_{tot} \rangle = \langle I_1 \rangle + \langle I_2 \rangle + \langle I_{12} \rangle \qquad \text{(I3')}$$

What are the conditions for a non-zero interference term ? Three conditions should be accomplished for the two waves:

- polarization along the same direction (at least to a certain extent)
- same frequency
- constant phase shift.

These are called *coherence conditions*. Such perfect coherence is hard to find in practice, but one can relax the above conditions for *partial coherent waves*.

3. Interference of two harmonic plane waves

The electric fields are parallel, so we need no more vectors:

$$E_1 = E_{01} \exp\left[i\left(\omega t - \vec{k}_1 \cdot \vec{r}_1 + \varphi_1\right)\right]$$
$$E_2 = E_{02} \exp\left[i\left(\omega t - \vec{k}_2 \cdot \vec{r}_2 + \varphi_2\right)\right]$$

The interference term is

$$I_{12} = \frac{1}{2\zeta} 2\sqrt{|E_{01}|^2 |E_{02}|^2} \cos\left[\left(\vec{k}_1 \cdot \vec{r}_1 - \vec{k}_2 \cdot \vec{r}_2\right) + \varphi_1 - \varphi_2\right]$$

The two waves propagate in the same material, hence $k_1 = k_2 = \frac{n\omega}{c}$. In optical experiments the propagation directions of both waves are almost the same, so the wave-vectors are equal as vectors:

$$\left(\vec{k}_1 \cdot \vec{r}_1 - \vec{k}_2 \cdot \vec{r}_2\right) \cong k\Delta r = \frac{n\Delta r}{\omega / c} = \frac{2\pi}{\lambda_0} n\Delta r \qquad \text{(I4)}$$

where λ_0 is the wavelength in vacuum. In (I4) is apparent *the optical path difference* obtained by multiplying the path difference by the refractive index:

$$[\Delta r] = n\Delta r \qquad \text{(I5)}$$

Eq. (I3') is (the time average does not change the cos function):

$$\langle I_{tot} \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\sqrt{\langle I_1 \rangle \langle I_2 \rangle} \cos(k\Delta r + \varphi_1 - \varphi_2) \qquad \text{(I6)}$$

The intensity has a sinusoidal spatial variation between maxima and minima of intensity

$$\langle I_{\max} \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\sqrt{\langle I_1 \rangle \langle I_2 \rangle}, \qquad \text{for } k\Delta r + \varphi_1 - \varphi_2 = 2m\pi \qquad \text{(I7')}$$

$$\langle I_{\min} \rangle = \langle I_1 \rangle + \langle I_2 \rangle - 2\sqrt{\langle I_1 \rangle \langle I_2 \rangle}, \qquad \text{for } k\Delta r + \varphi_1 - \varphi_2 = (2m+1)\pi \qquad \text{(I7'')}$$

The stationary interference pattern consists of a succession of surfaces definded by the condition that the phase shift should be an even number of π's for maxima and an odd number of π's for minima. Using (I4) the optical path difference is (for $\varphi_1 - \varphi_2 = 0$):

$$[\Delta r] = n\Delta r = 2m\frac{\lambda_0}{2} \qquad \text{for maxima} \qquad \text{(I8')}$$

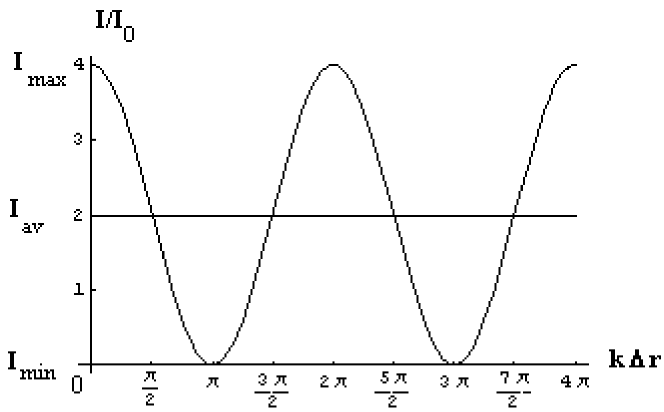
$$[\Delta r] = n\Delta r = (2m+1)\frac{\lambda_0}{2} \qquad \text{for minima} \qquad \text{(I8'')}$$

If the phase difference $\varphi_1 - \varphi_2$ has a random variation with time, maxima and minima are not observed because the interference term is averaged out.

Question: Why the phase difference $\varphi_1 - \varphi_2$ could have a random variation with time ? *Hint*: think of the atoms generating light.

Particular case: equal intensities $I_1 = I_2 = I_0$, $\varphi_1 - \varphi_2 = 0$;

$I_{\max} = 4I_0$, $I_{\min} = 0$. The total intensity varies as in the following figure. The average intensity is $2I_0$.



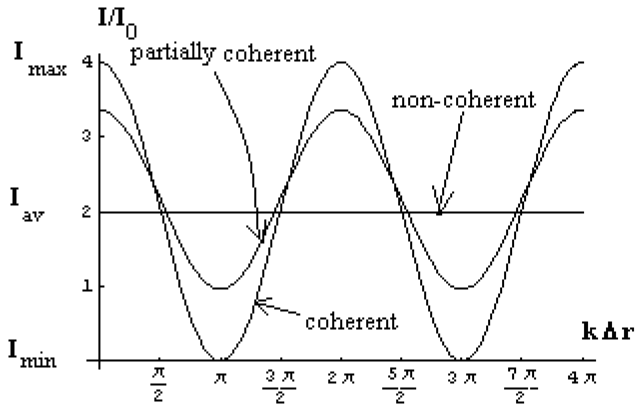
The visibility of the interference pattern is defined as

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \tag{I9}$$

For the above situation V=1

Question: Draw the intensity pattern for $I_1 = 2I_0$, $I_2 = I_0$ and compute the visibility.

Remark. Real beams waves are not harmonic (or monochromatic), but have several components of different frequencies, which form the *frequency band* of the beam. The waves are *partially coherent* and the visibility is always smaller than for the perfect coherent waves. See the figure below for the possible situations. In all three cases the average intensity is $2I_0$.



4. The importance of interference

Interferential devices are the basis of many of the most accurate and sensitive measurements made in science. Their uniqueness originates in the phase dependence on the path difference, namely in the spatial term $\exp[i(\omega t - k\Delta r)]$: $k\Delta r = 2\pi \frac{n\Delta r}{\lambda_0}$. In the visible domain $\lambda \approx 5 \cdot 10^{-7}$ m, so for a path difference of just one micrometer a 4π phase shift becomes visible. But we are able to easily measure phase shifts and fringe shifts corresponding to just $\pi/100$, hence to measure $\Delta r \approx \lambda/200 \approx 2.5\,\text{nm}$. With special care the resolution could diminish 10 times or even more. The use of such small path differences is rather impressive. Any physical quantity which can be linked to a path difference may be measured with high accuracy. What are these quantities? Length, time, wavelength, but also those which change the refractive index: temperature, chemical composition, electric and magnetic fields and many others.