# 2.2. MECANICA ANALITICA 2. ECUATIILE LUI HAMILTON

## 2.2.1. Introducere. Impulsuri si forte generalizate. Spatiul fazelor

In ultima parte a §1.2.7 am introdus impulsurile generalizate prin definitia (2.1.17):

$$\boxed{\frac{\partial L}{\partial \dot{q}_\alpha} \equiv p_\alpha} \qquad (2.1.17)$$

si fortele generalizate

$$\boxed{\frac{\partial L}{\partial q_\alpha} = Q_\alpha} \qquad (2.1.18)$$

In aceste ecuatii variabilele independente sunt coordonatele generalizate $q_1, q_2, ..., q_\alpha$ si vitezele generalizate $\dot{q}_1, \dot{q}_2, ..., \dot{q}_\alpha$. Hamilton a introdus o alta metoda, folosind ca variabile independente cele $f$ coordonatele generalizate $q_1, q_2, ..., q_\alpha$ si cele $f$ impulsuri generalizate $p_1, p_2, ..., p_f$ definite prin (2.1.17). Acesti **parametri** se numesc **variabile canonice** iar fiecare pereche $\{q_\alpha, p_\alpha\}$ se numeste **pereche de variabile conjugate**.

Starea sistemului se poate descrie printr-*un punct unic* intr-un spatiu cu 2*f* dimensiuni, ($q_1, q_2, ..., q_\alpha, p_1, p_2, ..., p_f$), numit **spatiul fazelor si notat cu $\Gamma$.** Evolutia in timp a sistemului nostru este reprezentata de *traiectoria acestui punct* in *spatiul* $\Gamma$.

*Exemplu.*

Pentru un oscilator armonic 1D, pozitia este data de $x(t) = a\sin(\omega t + \varphi_0)$, cu *a, ω* si $\varphi$ constante iar impulsul este $p(t) = m\dot{x}(t) = ma\omega\cos(\omega t + \varphi_0)$. Spatiul $\Gamma$ are 2 dimensiuni, iar traiectoria in acest spatiu se obtine eliminand timpul. Rezultatul este *o elipsa*

$$\frac{x^2}{a^2} + \frac{p^2}{(ma\omega)^2} = 1$$

reprezentata in figura urmatoare

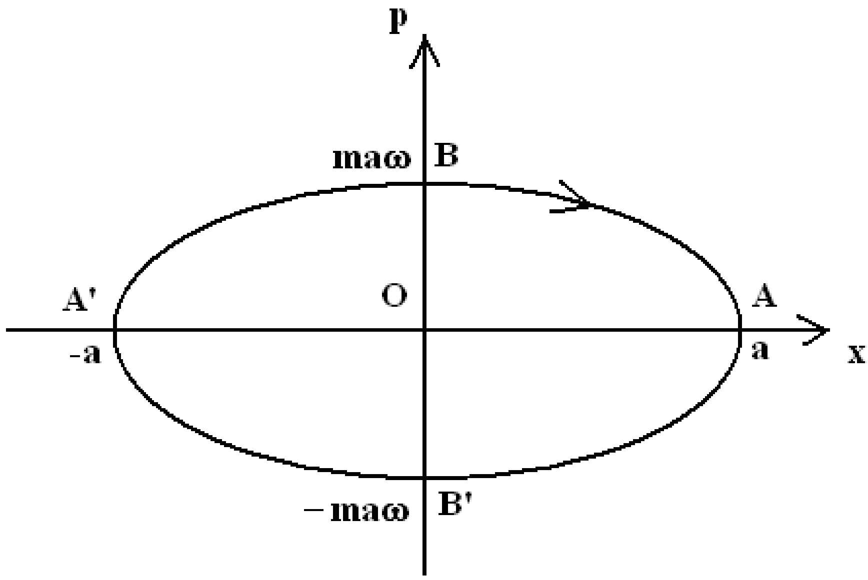


Fig. 1. Traiectoria unui oscilator armonic liniar in spatiul fazelor

Punctul reprezentativ, numit "*faza*" se deplaseaza mereu de-a lungul elipsei, deoarece energia totala se conserva. In punctele A si A' corpul atinge elongatiile maxime, $x(\mathrm{A\,sau\,A'}) = \pm a$ si are impulsul nul. In punctele B si B' corpul trece prin x=0 cu impuls maxim $p(\mathrm{B\,sau\,B'}) = \pm ma\omega$.

### 2.2.2 Ecuatiile canonice ale lui Hamilton

Ecuatiile Lagrange nu contin impulsuri generalizate. Introducem o noua functie, numita **fuctia Hamilton** sau **Hamiltonian** print-o *transformare Legendre*:

$$\boxed{H(p,q,t) = \sum_{\alpha=1}^{f} p_\alpha \dot{q}_\alpha - L(q,\dot{q},t)} \qquad (2.2.1)$$

Vom vedea mai tarziu ca **in cazuri speciale Hamiltonianul este constant si se identifica cu energia sistemului.**

Vom arata ca transformarea (2.2.1) elimina vitezele generalizate si introduce impulsurile generalizate. Sa incepem cu cazul $f$=1. Diferentiala lui $H$ este:

$$\mathrm{d}H = \dot{q}\mathrm{d}p + p\mathrm{d}\dot{q} - \mathrm{d}L$$

Dar $\mathrm{d}L(q,\dot{q},t) = \frac{\partial L}{\partial q}\mathrm{d}q + \frac{\partial L}{\partial \dot{q}}\mathrm{d}\dot{q} + \frac{\partial L}{\partial t}\mathrm{d}t = \dot{p}\mathrm{d}q + p\mathrm{d}\dot{q} + \frac{\partial L}{\partial t}\mathrm{d}t$. Inlocuim in $\mathrm{d}H$ si reducem termenii subliniati:

$$\mathrm{d}H = \dot{q}\mathrm{d}p + \underline{p\mathrm{d}\dot{q}} - \dot{p}\mathrm{d}q \underline{- p\mathrm{d}\dot{q}} - \frac{\partial L}{\partial t}\mathrm{d}t = \dot{q}\mathrm{d}p - \dot{p}\mathrm{d}q - \frac{\partial L}{\partial t}\mathrm{d}t$$

Deci $H$ depinde numai de $p$, $q$, si de $t$:

$$dH = \frac{\partial H}{\partial q}\mathrm{d}q + \frac{\partial H}{\partial p}\mathrm{d}p + \frac{\partial \mathrm{H}}{\partial \mathrm{t}}\mathrm{d}t = \dot{q}\mathrm{d}p - \dot{p}\mathrm{d}q - \frac{\partial L}{\partial t}\mathrm{d}t = \tag{2.2.2}$$

Prin identificare gasim un sistem simetric, care este un caz particular al ecuatiilor canonice pentru $f$=1:

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{2.2.3}$$

precum si o egalitate pentru derivate partiale in timp:

$$\frac{\partial \mathrm{H}}{\partial \mathrm{t}} = -\frac{\partial L}{\partial t} \tag{2.2.4}$$

Pentru $f$ grade de libertate obtinem:

$$\mathrm{d}H = \sum_{\alpha=1}^{f}\dot{q}_\alpha \mathrm{d}p_\alpha + \sum_{\alpha=1}^{f} p_\alpha \mathrm{d}\dot{q}_\alpha - \mathrm{d}L(q,\dot{q},t) =$$

$$\sum_{\alpha=1}^{f}\dot{q}_\alpha \mathrm{d}p_\alpha - \sum_{\alpha=1}^{f}\dot{p}_\alpha \mathrm{d}q_\alpha - \frac{\partial L}{\partial t}\mathrm{d}t = \sum_{\alpha=1}^{f}\frac{\partial \mathrm{H}}{\partial p_\alpha}\mathrm{d}p_\alpha + \sum_{\alpha=1}^{f}\frac{\partial \mathrm{H}}{\partial q_\alpha}\mathrm{d}q_\alpha + \frac{\partial H}{\partial t}\mathrm{d}t \tag{2.2.5}$$

Prin identificare gasim **ecuatiile canonice**:

$$\boxed{\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}} \qquad \boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}} \qquad \alpha = 1,2,\dots,f \tag{2.2.6}$$

**Aceste 2*f* ecuatii diferentiale de ordinul I sunt complet echivalente cu cele *f* ecuatii de ordinul II ale lui Lagrange**.

*Exemplu*. Reluam oscilatorul liniar armonic.

$T = \frac{p^2}{2m}$, $U = \frac{kx^2}{2}$, $H = \frac{p^2}{2m} + \frac{kx^2}{2}$. Ecuatiile canonice sunt

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial x} = -kx.$$

Descrierile Lagrange si Hamilton sunt echivalente si, in acest caz, coincid cu ecuatia Newton. Insa **pentru sisteme cu legaturi descrierile Lagrange sau Hamilton sunt mai generale si mai simple. De ce ?**

### 2.2.3. Paranteze Poisson ► ► ►

**Definitie.** Fie doua functii de varibilele canonice si de timp $\varphi(p_\alpha, q_\alpha, t)$ si $\psi(p_\alpha, q_\alpha, t)$. **Paranteza Poisson** (pP) a celor doua functii este

$$[\varphi,\psi] = \sum_{\alpha=1}^{f}\left(\frac{\partial \varphi}{\partial p_\alpha}\frac{\partial \psi}{\partial q_\alpha} - \frac{\partial \psi}{\partial q_\alpha}\frac{\partial \varphi}{\partial p_\alpha}\right) \tag{2.2.7}$$

*Proprietati*.

$$[\varphi_1 \cdot \varphi_2, \psi] = \varphi_1 \cdot [\varphi_1 \cdot, \psi] + [\varphi_1, \psi] \cdot \varphi_2 \tag{2.2.8}$$

$$\frac{\partial}{\partial t}[\varphi,\psi] = \left[\frac{\partial \varphi}{\partial t}, \psi\right] + \left[\varphi, \frac{\partial \psi}{\partial t}\right] \tag{2.2.9}$$

Daca una dintre functii este o varibila canonica

$$[\varphi, q_\beta] = \frac{\partial \varphi}{\partial p_\beta} \qquad [\varphi, p_\beta] = -\frac{\partial \varphi}{\partial q_\beta} \qquad (2.2.10)$$

Daca ambele sunt variabile canonice,

$$\boxed{[q_\beta, q_\gamma] = 0} \qquad \boxed{[p_\beta, p_\gamma] = 0} \qquad \boxed{[p_\beta, q_\gamma] = \delta_{\beta\gamma}} \qquad (2.2.11)$$

*Exercitiu*. Demonstrati ca

$$\boxed{\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial \varphi}{\partial t} + [H, \varphi]} \qquad (2.2.12)$$ ◄◄◄

Rezulta ca **daca o functie $\varphi$ nu depinde explicit de timp si pP sa cu Hamiltoniana este nula, functia este o <u>constanta a miscarii</u>**